Subnormal Series

This lab is a continuation of the work you did with the other quotient group labs. You may want to review those labs before attempting this one. In these labs you considered chains of subgroups of the form $G \geq H \geq \{e\}$; where e is the identity of G. Note that chains of this form have length 2. You may wonder what happens if one considers longer chains of subgroups of the form $G \geq H_1 \geq H_2 \geq \cdots \geq H_n \geq \{e\}$. We will consider such chains in this lesson. Note that if H_{i+1} is a normal subgroup of H_i for $i = 1, 2, \ldots, n-1$ and H_1 is a normal subgroup of G, then we may construct quotient groups $G/H_1, H_1/H_2, \ldots H_{n-1}/H_n$, and $H_n/\{e\} \cong H_n$. In this case the chain is called a subnormal series and written $G \triangleright H_1 \triangleright H_2 \triangleright \ldots \triangleright H_n \triangleright \{e\}$ (in general, $H \triangleleft G$ means H is a normal subgroup of G. For our first example, we will consider the subnormal series $\mathbb{Z}_8 \triangleright \{0, 2, 4, 6\} \triangleright \{0, 4\} \triangleright \{0\}$ (since $\mathbb{Z}_8$ is abelian, we know that each subgroup in the chain is a normal subgroup of the proceeding group).

Exercises:

1. Using the PascGalois JE program draw the first 64 rows of Pascal's triangle mod 8. Next consider the quotient group $\mathbb{Z}_8/\{0, 2, 4, 6\}$. Using the color options in the PascGalois JE program to recolor the elements of $\mathbb{Z}_8$ by identifying elements in a common coset with the same color. Redraw the triangle and describe the picture that you obtain.

2. At this point we would like to get an image of the quotient group $\{0, 2, 4, 6\}/\{0, 4\}$. There are a several ways we can do that with the PascGalois JE program but probably the easiest method is to start with the image of the subgroup $\{0, 2, 4, 6\}$ of $\mathbb{Z}_8$. Since $\{0, 2, 4, 6\} = \langle 2 \rangle$ we can view this subgroup by using 2 as the seed instead of the 1. Once this is graphed we do the coset coloring for the subgroup $\{0, 4\}$. Redraw the triangle and describe the picture that you obtain.

3. The third and final quotient group we get from this subnormal series is $\{0, 4\}/\{0\} \cong \{0, 4\}$. Now it does not take a rocket scientist to figure out what this group is but we will nonetheless create its image. Since $\{0, 4\} = \langle 4 \rangle$ we can view this subgroup by using 4 as the seed. Create this triangle and describe the picture that you obtain.

4. The next example we will consider is the dihedral group $\mathbb{D}_4$ the symmetry group of a square. You may want to review the labs regarding dihedral groups. Using the PascGalois JE program draw the first 64 rows of $\mathbb{D}_4$ using the generators $R1$ and $F0$. Recall that in the notation of the PascGalois JE program $R1$ represents a counter-clockwise rotation by 90° and $F0$ is a flip (or reflection) over the horizontal.

   Note that $H_1 = \{R_0, R_1, R_2, R_3\}$ is a normal subgroup of $\mathbb{D}_4$. Why? So $\mathbb{D}_4/H_1$ is a quotient group. What is the order of this group? What are the cosets? Which coset acts as the identity? Using the color options in PascGalois JE, recolor the elements of $\mathbb{D}_4$ by identifying elements in a common coset with the same color. Describe the picture that you obtain. Have you seen it before?

5. Now consider the subgroup $H_2 = \{R_0, R_2\}$. Explain why the chain $\mathbb{D}_4 \triangleright H_1 \triangleright H_2 \triangleright \{0\}$ is a subnormal series. We have already considered the quotient $\mathbb{D}_4/H_1$ in the previous exercise. Get an image of the group H_1/H_2, recall that you need to know the generators of H_1. Describe the picture that you obtain.

6. Consider the subgroup $H = \{R_0, R_2\}$ of D_4. You should check that this is indeed a normal subgroup and that $\mathbb{D}_4 \triangleright H \triangleright \{0\}$ is a subnormal series. Determine the cosets of $\mathbb{D}_4/H$ and generate the image of the group. This quotient group is isomorphic to a group you have seen before. Which group is it?

7. Now let's consider $\mathbb{D}_8$, the symmetry group of a regular octagon. Find a subnormal series $\mathbb{D}_8 \triangleright H_1 \triangleright H_2 \triangleright H_3 \triangleright \{0\}$ where $|H_1| = 8$, $|H_2| = 4$, and $|H_3| = 2$. Perform the same type of analysis that you did for $\mathbb{D}_4$ in the previous exercise. Do you obtain similar results?

8. Can you find a normal subgroup $H \triangleleft \mathbb{D}_8$ such that $\mathbb{D}_8/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (the Klien-4 group)? If so, what is H? Create the triangle for this quotient group.