Self-Similarity and the Klein-4 Group

Recall that the Klein-4 Group is the noncyclic of order 4 that can be represented with the elements $\{a, b, c, d\}$ using the following operation table.

·	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

In this lab we will characterize the self-similar growth structure of $P_{b,c}$. Recall that $P_{b,c}$ is our notation for the PascGalois triangle that has the element b down the left hand side and c down the right hand side. The self-similarity studied in this lab is different from that of fractal geometry. Instead of "zooming in" to see copies of a pattern, we "zoom out". Before starting the exercises you should construct the subgroup lattice for the Klein-4 group.

Exercises:

1. The first thing we need to do is to tell the PascGalois JE program that we wish to work with the Klein-4 group. Under the Group tab you will notice that there is no option for the Klein-4 group. To input a group structure that is not one of the built-in groups we select the Gn (User Defined Group) type. You will notice that when you do a table will appear at the bottom of the window and some other options will be above this table. This group type allows you to specify a group structure by inputing the group's operation table. In general, the operation table you supply does not need to form a group. In fact, some rather interesting images can be formed by non-group structures.

   Once you have selected the Gn (User Defined Group) type, change the group order to 4 and for the group name input something like "Klein-4". In the table, the first column is for the elements of the group. Input a, b, c and d into this first column. You will note that these elements are automatically placed across the top of the table as well. Now fill in the body of the table so that it matches the operation table of the Klein-4 group.

   Now we will construct $P_{b,c}$ as described above. Click on the 1-D Automaton tab, you should note that the default element is a. This means that the program found that the identity element of the group was a, which you should have noticed as well. If you are using an older version of the PascGalois JE program the default element may not have automatically changed, in this case click on the Use Group Identity button and the default element should change to a. If you get an error then you should go back to the group table and make sure that you input the elements in the correct positions. Now click on the Seed tab, increase the number of columns to 2 and input b in the first cell and c in the second cell.

   Now we will graph $P_{b,c}$ using different numbers of rows. Try the first 4, 8, 16, 32, and 64 rows of the triangle. What is the dominant pattern emerging? What sub-triangles of the triangle are repeated at different scales?

2. Give a description of the first 2^n rows of $P_{b,c}$ that applies for any $n \geq 3$. Draw a diagram that provides the overall structure as part of your answer. This diagram is called a growth triangle — it reveals the self-similar structure of the PascGalois triangle. How does the group structure determine this growth triangle? Note: You should pay close attention to the boundaries of the sub-triangles as you try to understand the growth triangle in terms of the structure of the Klein-4 group. It may also help to write out the first 16 rows by hand to understand how the geometry of the growth triangle is determined by the group multiplication.

3. Determine the growth triangles for $P_{1,1}$ of $\mathbb{Z}_n$ for $n = 2, 3, 4, 5$ and 6. Compare the patterns you see with the subgroup lattice of each group. Then compare the orders of the group elements for each group versus the number of rows needed to repeat the self-similar structure. Does the value of n, whether it is prime or composite, seem to affect the corresponding PascGalois triangle? What about the number of relatively prime factors? (The answers to these last two questions are addressed in more detail in the projects on quotient groups).