Exam #1 Key

1. (*10 Points Each*) Answer the following questions on languages and grammars.

   (a) Prove that for every two languages L_1 and L_2 that $(L_1 L_2)^R = L_2^R L_1^R$.

   Solution: Let L_1 and L_2 be any languages and let w_1 and w_2 be arbitrary elements of L_1 and L_2 respectively. Denote $w_1 = a_1 a_2 \cdots a_n$ and $w_2 = b_1 b_2 \cdots b_m$ where a_i and b_i are takes from the alphabet Σ. Let $w \in (L_1 L_2)^R$, then w has the form $(w_1 w_2)^R$, and

   $$(w_1 w_2)^R = (a_1 a_2 \cdots a_n b_1 b_2 \cdots b_m)^R = b_m \cdots b_2 b_1 a_n \cdots a_2 a_1 = w_2^R w_1^R \in L_2^R L_1^R$$

   Hence, $(L_1 L_2)^R \subseteq L_2^R L_1^R$. Reversing the argument, assume that $w \in L_2^R L_1^R$, then it has the form $w = w_2^R w_1^R$. The chain of equalities above establishes that in general $w_2^R w_1^R = (w_1 w_2)^R$, hence $w \in (L_1 L_2)^R$, giving $L_2^R L_1^R \subseteq (L_1 L_2)^R$, and finally, $(L_1 L_2)^R = L_2^R L_1^R$.

   (b) Give a grammar for the language of all non-empty palindromes.
   Solution:

$$
\begin{aligned}
S &\longrightarrow aAa \\
S &\longrightarrow bAb \\
S &\longrightarrow a \\
S &\longrightarrow b \\
A &\longrightarrow aAa \\
A &\longrightarrow bAb \\
A &\longrightarrow a \\
A &\longrightarrow b \\
A &\longrightarrow \lambda
\end{aligned}
$$

2. (*10 Points Each*) For each of the following languages, give a regular expression for that language.
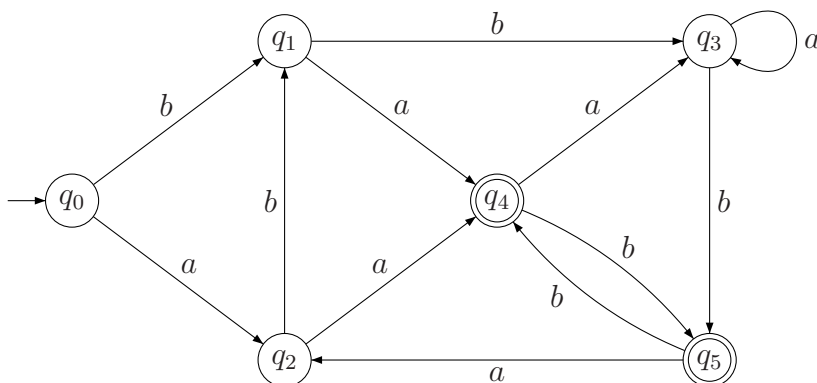
   (a) $L = \{vwv \mid v, w \in \{a, b\}^*, |v| = 2\}$
   Solution: $aa(a + b)^* aa + ab(a + b)^* ab + ba(a + b)^* ba + bb(a + b)^* bb$.

   (b) $L \subset \{0, 1\}^*$, is the language where each word contains an even number of 0's.
   Solution: $(1^* 0 1^* 0 1^*)^*$.

3. (*35 Points*) Consider the following DFA, A.

(a) Determine if the automaton accepts the following words. Display the sequence of states for each word.

    i. $aabbaa$ — $q_0 \to q_2 \to q_4 \to q_5 \to q_4 \to q_3 \to q_3$ — Not Accepted

    ii. $bbaabbaa$ — $q_0 \to q_1 \to q_3 \to q_3 \to q_3 \to q_5 \to q_4 \to q_3 \to q_3$ — Not Accepted

    iii. $ababab$ — $q_0 \to q_2 \to q_1 \to q_4 \to q_5 \to q_2 \to q_1$ — Not Accepted

(b) Is $L((aabb)^*) \subseteq L(A)$? Why or why not? If not, what is the largest subset of $L((aabb)^*)$ that is a subset of $L(A)$?

Solution: $L((aabb)^*) \not\subseteq L(A)$ since $\lambda \in L((aabb)^*)$ and $\lambda \notin L(A)$. The largest subset of $L((aabb)^*)$ that is contained in $L(A)$ is $L((aabb)^+)$. The word $aabb$ drives A to q_4, a final state. Once at q_4, $aabb$ drives A back to q_4. Hence any positive number of repetitions of the string $aabb$ drives A to q_4, and hence the word is accepted.

(c) Fill in the blank with the appropriate condition(s) $\{b^m a^n \mid \underline{\hspace{3cm}}\} \subseteq L(A)$. Justify your answer.

Solution: ($m = 0$ and $n = 2$) or ($m = 1$ and $n = 1$) or ($m \geq 3$ and $n = 0$) or ($m \geq 3, m$ odd and $n = 2$). Clearly aa and ba are accepted, which takes care of the first two cases. If the word consists of just b's then after three b's you are at q_5, any more b's drives the word to either states q_4 or q_5, both final. This takes care of the third case. If any a's follow a string of b's of length greater than or equal to 3 then the a's are read starting from either q_4 or q_5. From q_4 any number of a's will end at q_3 (non-final). But to start reading a's at q_4 means that an even number of b's were read. From q_5, two a's must be read to end at q_4, any more will result in ending at q_3 and less at q_2. To start reading a's at q_5 means that an odd number of b's had already been read.

4. (*35 Points*) Consider the following NFA, A.
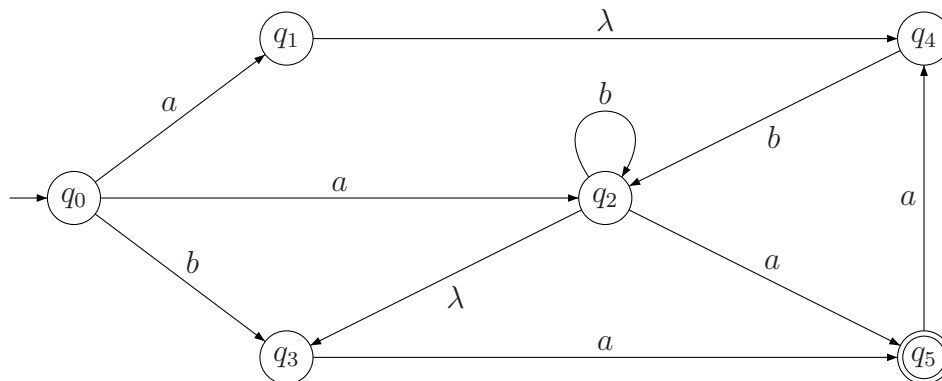
(a) Determine if the automaton accepts the following words. If it does, display the sequence of states that drive the word to a final state.

   i. *abab* — Not Accepted

   ii. *abbbaaba* — $q_0 \to q_2 \to q_2 \to q_2 \to q_2 \to q_5 \to q_4 \to q_2 \to q_5$ — Accepted

   iii. *bbaab* — Not Accepted

(b) What is the largest run of a's in any accepted word? List all of the words with that number of runs.

   Solution: 3, $\{aa(ab^n a)^m \mid n, m \geq 1\}$.

(c) What is the largest number of a's that can end any accepted word of length at least 5? Justify your answer.

   Solution: There can be at most one a ending any accepted word if the length of the word is at least 5. Any, and every, accepted word of length greater then two must at some point pass through state q_2. Furthermore, any word of this type must either use the b loop at q_2 or the b edge from q_4 to q_2. From q_2 there are only two non-loop pathways to q_5, both read a single a.

(d) Convert this NFA to a DFA.

   Solution:

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