1. **True & False:** (*2 Points Each*) Mark each of the following as being either true or false.

   (a) **FALSE:** The union of two deterministic context-free languages is deterministic context-free.

   (b) **TRUE:** The star closure of any context free language can be realized as a push-down automaton.

   (c) **TRUE:** If L_1 is context free and L_2 is regular then the language $L = L_1 - L_2$ is context-free.

   (d) **FALSE:** The intersection of two context-free languages is context-free.

   (e) **FALSE:** The complement of a context free language is context free.

2. (*15 Points*) State Turing's Thesis (also known as The Church-Turing Thesis) and explain its meaning.

   Solution: Turing's Thesis is a hypothesis that states that any computation that can be carried out by mechanical means can be performed by some Turing machine.

   This gives us a mathematical model for all possible mechanical computations. An algorithm is thus defined in a similar manner, as a process that can be completed by a Turing machine. Using this, we can determine if there are functions that cannot be computed simply by showing that there is no Turing machine that can compute it.

3. (*15 Points*) Construct an NPDA for the language $L = \{a^n b^m \mid n \leq m \leq 3n, n \geq 0\}$.

   Solution: State q_f is the only final state.

$$\begin{aligned}
\delta(q_0, \lambda, z) &= \{(q_f, z)\} \\
\delta(q_0, a, z) &= \{(q_1, az), (q_1, aaz), (q_1, aaaz)\} \\
\delta(q_1, a, a) &= \{(q_1, aa), (q_1, aaa), (q_1, aaaa)\} \\
\delta(q_1, b, a) &= \{(q_2, \lambda)\} \\
\delta(q_2, b, a) &= \{(q_2, \lambda)\} \\
\delta(q_2, \lambda, z) &= \{(q_f, z)\}
\end{aligned}$$

4. (*15 Points*) Show that the language $L = \{a^n b^n \mid n \geq 1\} \cup \{a\}$ is deterministic context free.

   Solution: State q_f is the only final state.

$$\begin{aligned}
\delta(q_0, a, z) &= \{(q_f, az)\} \\
\delta(q_f, a, a) &= \{(q_1, aa)\} \\
\delta(q_f, b, a) &= \{(q_2, \lambda)\} \\
\delta(q_1, a, a) &= \{(q_1, aa)\} \\
\delta(q_1, b, a) &= \{(q_2, \lambda)\} \\
\delta(q_2, b, a) &= \{(q_2, \lambda)\} \\
\delta(q_2, \lambda, z) &= \{(q_f, z)\}
\end{aligned}$$

5. (*15 Points*) Create a context free grammar for the language

$$L = \{a^n b^m c^{n+m} \mid n \geq 0, m \geq 0\}$$

and then convert the grammar to an NPDA.

Solution:

$$\begin{aligned} S &\rightarrow aSc \mid B \\ B &\rightarrow bBc \mid \lambda \end{aligned}$$

Alter the form for conversion

$$\begin{aligned} S &\rightarrow aSC \mid B \\ B &\rightarrow bBC \mid \lambda \\ C &\rightarrow c \end{aligned}$$

Convert

$$\begin{aligned} \delta(q_0, \lambda, z) &= \{(q_1, Sz)\} \\ \delta(q_1, \lambda, z) &= \{(q_f, z)\} \\ \delta(q_1, a, S) &= \{(q_1, SC)\} \\ \delta(q_1, \lambda, S) &= \{(q_1, B)\} \\ \delta(q_1, b, B) &= \{(q_1, BC)\} \\ \delta(q_1, \lambda, B) &= \{(q_1, \lambda)\} \\ \delta(q_1, c, C) &= \{(q_1, \lambda)\} \end{aligned}$$

6. (*20 Points*) Show that the language $L = \{w \mid n_a(w) \cdot n_b(w) = n_c(w)\}$ is not context free.

Solution: By way of contradiction, assume that L is context-free. Then there exists a positive integer m such that for any word $w \in L$ with $|w| \geq m$, we can write $w = uvxyz$, with $|vxy| \leq m$ and $|vy| \geq 1$, such that $uv^ixy^iz \in L$ for all $i = 0, 1, 2, 3, \ldots$. Let $w = a^m b^m c^{m^2}$. Note that $w \in L$ and $|w| = m^2 + 2m \geq m$, so we can write $w = uvxyz$ with the above conditions holding. This will give us five cases for vy, these are a^t, $a^t b^r$, b^t, $b^t c^r$, and c^t, where $t > 0$ and $r > 0$.

Case 1: If $vy = a^t$, then if we pump in one copy of v and y we have $m + t$ a's, so $n_a(w) \cdot n_b(w) = (m + t) \cdot m \neq m^2 = n_c(w)$, so $w_2 \notin L$.

Case 2: If $vy = a^t b^r$, then if we pump in one copy of v and y we have $m + t$ a's and $m + r$ b's, so $n_a(w) \cdot n_b(w) = (m + t) \cdot (m + r) \neq m^2 = n_c(w)$, so $w_2 \notin L$.

Case 3: If $vy = b^t$, then if we pump in one copy of v and y we have $m + t$ b's, so $n_a(w) \cdot n_b(w) = m \cdot (m + t) \neq m^2 = n_c(w)$, so $w_2 \notin L$.

Case 4: If $vy = b^t c^r$, then if we pump in one copy of v and y we have $m + t$ b's and $m^2 + r$ c's, so $n_a(w) \cdot n_b(w) = m \cdot (m + t) = m^2 + mt \geq m^2 + m > m^2 + r = n_c(w)$, so $w_2 \notin L$. Note that since $t > 0$, we have $mt \geq m$ and in addition since $t + r \leq m$ we have that $r < m$.

Case 5: If $vy = c^t$, then if we pump in one copy of v and y we have $m^2 + t$ c's, so $n_a(w) \cdot n_b(w) = m^2 \neq m^2 + t = n_c(w)$, so $w_2 \notin L$.

Since all cases have led to a contradiction, the language $L = \{w \mid n_a(w) \cdot n_b(w) = n_c(w)\}$ is not context-free.

7. (*20 Points*) Create a Turing machine by displaying its set of transitions that will take a binary number on the tape and add one to it. It assumed that there is a number on the tape and that the read/write head is on the far right digit of the number. The machine is to add one to the number on the tape (in binary) and return the read/write head to the far right digit of the number.

Solution:

$$
\begin{aligned}
\delta(q_0, 0) &= (q_1, 1, R) \\
\delta(q_0, 1) &= (q_0, 0, L) \\
\delta(q_0, \square) &= (q_1, 1, R) \\
\delta(q_1, 0) &= (q_1, 0, R) \\
\delta(q_1, 1) &= (q_1, 1, R) \\
\delta(q_1, \square) &= (q_f, \square, L)
\end{aligned}
$$