Spring 2012

Exam #2 Key

Math 362

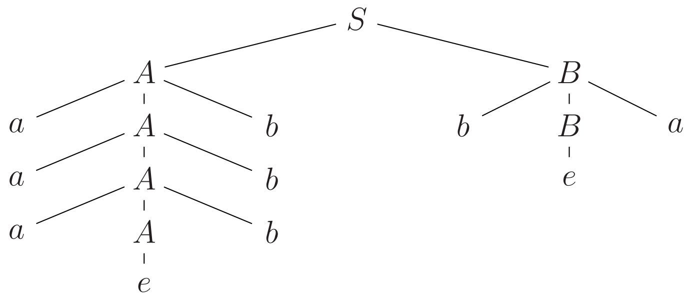
1. **Grammars and Parsing:** (*25 Points*) Consider the context-free grammar G with $\Sigma = \{a, b\}$ and rules
   - (1) $S \longrightarrow AB$
   - (2) $S \longrightarrow BA$
   - (3) $A \longrightarrow aAb$
   - (4) $B \longrightarrow bBa$
   - (5) $B \longrightarrow e$
   - (6) $A \longrightarrow e$

   (a) Construct a left-most derivation for the word $aaabbbba$. Put the rule number over the arrow at each step.

   Solution: $S \xrightarrow{(1)} AB \xrightarrow{(3)} aAbB \xrightarrow{(3)} aaAbbB \xrightarrow{(3)} aaaAbbbB \xrightarrow{(6)} aaabbbB \xrightarrow{(4)} aaabbbbBa \xrightarrow{(5)} aaabbbba$

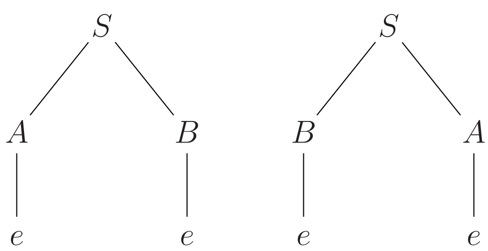
   (b) Construct the parse tree for your derivation of $aaabbbba$.

   Solution:



   (c) Is the language $L(G)$ ambiguous? If so, prove it.

   Solution: Yes, e has two different parse trees.



   (d) Is $L(G)$ regular? If so write $L(G)$ as a regular expression and if not describe $L(G)$ in words.

   Solution: No, $L(G)$ is not regular, $L(G) = \{w \mid w = a^n b^{n+m} a^m\} \cup \{w \mid w = b^n a^{n+m} b^m\}$

2. **Push-Down Automata and Determinism:** (*25 Points*) Consider the following PDA, A. A has starting state s, accepts by empty stack, $\Sigma = \{a, b, c\}$ and $\Gamma = \{a, b, d\}$

(1) $((s, e, e), (q, d))$
(2) $((q, a, e), (q, b))$
(3) $((q, b, e), (q, a))$
(4) $((q, c, e), (t, e))$
(5) $((t, a, a), (t, e))$
(6) $((t, b, b), (t, e))$
(7) $((t, e, d), (r, e))$
(8) $((r, a, e), (r, e))$
(9) $((r, b, e), (r, e))$

(a) Use a transition/state/input/stack table (as done in class) to test the strings $aabcabbabab$ and $aabcaab$.

Solution: $aabcabbabab$, is accepted,

Transition	State	Input	Stack
–	s	$aabcabbabab$	e
1	q	$aabcabbabab$	d
2	q	$abcabbabab$	bd
2	q	$bcabbabab$	bbd
3	q	$cabbabab$	$abbd$
4	t	$abbabab$	$abbd$
5	t	$bbabab$	bbd
6	t	$babab$	bd
6	t	$abab$	d
7	r	$abab$	e
8	r	bab	e
9	r	ab	e
8	r	b	e
9	r	e	e

and $aabcaab$, is not accepted,

Transition	State	Input	Stack
–	s	$aabcaab$	e
1	q	$aabcaab$	d
2	q	$abcaab$	bd
2	q	$bcaab$	bbd
3	q	$caab$	$abbd$
4	t	aab	$abbd$
5	t	ab	bbd

(b) Describe $L(A)$.

Solution:

$L(A) = \{wcw'u \mid u \in \Sigma^*$ and w' is the reverse of w with a's and b's interchanged$\}$

(c) Is the automaton deterministic? Why or why not?

Solution: Yes, the automaton is deterministic. There are no transitions of the form $((s, e, e), (q, e))$ and there are no compatible transitions.

3. **True & False:** (*20 Points*) Mark each of the following as being either true or false.

   (a) True — Any language that can be represented as the concatenation of a context-free language and a regular language can be accepted by a push-down automaton.

   (b) False — The intersection of two context-free languages is context-free.

   (c) True — The Kleene star of a context-free language is context-free.

   (d) False — The union of a context-free language with a regular language is regular.

   (e) True — The concatenation of a context-free language and a regular language is context-free.

   (f) True — The complement of a deterministic context-free language is deterministic context-free.

   (g) False — The complement of a context-free language can be represented as a finite union of context-free languages.

   (h) False — In order for a language to be non-context-free the alphabet of that language must contain at least 3 distinct characters.

   (i) True — The complement of a regular language is deterministic context-free.

   (j) True — The intersection of a context-free language and a regular language is context-free.

4. **Context-Free Languages:** (*20 Points*) Show that the language $L = \{a^k b^{2k} c^{3k} \mid k = 0, 1, 2, \ldots\}$ is not context-free.

   Solution: By way of contradiction assume that L is context-free. Then by the context-free pumping lemma we know that there exists a fixed positive integer n such that any word $w \in L$, with length $|w| \geq n$, can be written as the concatenation $w = uvxyz$ with vy not empty, $|vxy| \leq n$ and $uv^ixy^iz \in L$ for each $i \geq 0$. Let us choose the word $w = a^n b^{2n} c^{3n}$ where n is the positive integer guaranteed by the pumping lemma for language L. Since vy is not empty and $|vxy| \leq n$ we know that either $vy = a^j$, $vy = a^j b^r$, $vy = b^j$, $vy = b^j c^r$ or $vy = c^j$ where $0 < j, r \leq n$. We have the following cases,

   (a) If $vy = a^j$ then by the pumping lemma $uv^2xy^2z = a^{n+j} b^{2n} c^{3n} \in L$ which is absurd since $2(n+j) > 2n$.

   (b) If $vy = a^j b^r$ then by the pumping lemma $uv^2xy^2z = a^{n+j} b^{2n+r} c^{3n} \in L$ which is absurd since $3(n+j) > 3n$.

   (c) If $vy = b^j$ then by the pumping lemma $uv^2xy^2z = a^n b^{2n+j} c^{3n} \in L$ which is absurd since $2n + j > 2n$.

   (d) If $vy = b^j c^r$ then by the pumping lemma $uv^2xy^2z = a^n b^{2n+j} c^{3n+r} \in L$ which is absurd since $2n + j > 2n$.

   (e) If $vy = c^j$ then by the pumping lemma $uv^2xy^2z = a^n b^{2n} c^{3n+j} \in L$ which is absurd since $3n + j > 3n$.

   Hence, the language L is not context-free.

5. **Chomsky Normal Form:** (*20 Points*) Convert the following grammar to Chomsky Normal Form.

(1) $S \longrightarrow AB$
(2) $S \longrightarrow BA$
(3) $A \longrightarrow aAb$
(4) $B \longrightarrow bBa$
(5) $B \longrightarrow e$
(6) $A \longrightarrow e$

Solution: In the first algorithm we have $E = \{A, B, S\}$ so we add in

$S \longrightarrow A$
$S \longrightarrow B$
$A \longrightarrow ab$
$B \longrightarrow ba$

and remove 5 and 6. In the second algorithm we have $NT(A) = \{A\}$, $NT(B) = \{B\}$, and $NT(S) = \{S, A, B\}$. So we add in

$S \longrightarrow aAb$
$S \longrightarrow bBa$
$S \longrightarrow ab$
$S \longrightarrow ba$

and remove $S \longrightarrow A$ and $S \longrightarrow B$. In the final algorithm we keep 1 and 2 from the original list and replace the rest. Our final list of rules is

$S \longrightarrow AB$
$S \longrightarrow BA$
$A \longrightarrow X_a X_{Ab}$
$X_{Ab} \longrightarrow AX_b$
$X_a \longrightarrow a$
$X_b \longrightarrow b$
$B \longrightarrow X_b X_{Ba}$
$X_{Ba} \longrightarrow BX_a$
$A \longrightarrow X_a X_b$
$B \longrightarrow X_b X_a$
$S \longrightarrow X_a X_{Ab}$
$S \longrightarrow X_b X_{Ba}$
$S \longrightarrow X_a X_b$
$S \longrightarrow X_b X_a$