Exam #3 Key

1. **Short Answer:** (*5 Points Each*)**:** Answer all of the following.

   (a) Define countably infinite. — A set A is countably infinite if it can be put into a one-to-one correspondence with a subset of the natural numbers.

   (b) Define a partial Turing computable function. — This is a function f such that there exists a Turing machine M with $M(w) = f(w)$ if $w \in Dom(f)$ and $M(w) \uparrow$ if $w \notin Dom(f)$.

   (c) Define a decidable language. — This is a language L such that there exists a Turing machine with $M(w) = 1$ if $w \in L$ and $M(w) = 0$ if $w \notin L$.

   (d) Define a Turing enumerable language. — This is the same as a semi-decidable language. That is, there exists a Turing machine with $M(w) = 1$ if $w \in L$ and $M(w) \uparrow$ if $w \notin L$.

   (e) State the Church-Turing thesis. — The Church-Turing thesis states that every computer algorithm can be implemented as a Turing machine.

2. **Determinism:** (*25 Points*) Show that the language $L = \{wcw^R \mid w \in \{a,b\}^*\}$ is deterministic context-free.

   Solution: Using f as the favorable state we have the following transitions.

   (1) $((s,a,e),(s,a))$
   (2) $((s,b,e),(s,b))$
   (3) $((s,c,e),(q,e))$
   (4) $((q,a,a),(q,e))$
   (5) $((q,b,b),(q,e))$
   (6) $((q,\$,e),(f,e))$

3. **Turing Machines:** (*25 Points Each*)

   (a) Write a complete set of transitions for a Turing Machine that semidecides the language $L = \{wcw^R \mid w \in \{a,b\}^*\}$.

   Solution: Let h be the only halting state.

$((s,a),(q_a,\sqcup))$
$((s,b),(q_b,\sqcup))$
$((s,c),(x,\rightarrow))$
$((s,\sqcup),(t,\sqcup))$
$((s,\triangleright),(s,\rightarrow))$

$((q_a,a),(m_a,\rightarrow))$
$((q_a,b),(m_a,\rightarrow))$
$((q_a,c),(m_a,\rightarrow))$
$((q_a,\sqcup),(m_a,\rightarrow))$
$((q_a,\triangleright),(m_a,\rightarrow))$

$((m_a,a),(m_a,\rightarrow))$
$((m_a,b),(m_a,\rightarrow))$
$((m_a,c),(m_a,\rightarrow))$
$((m_a,\sqcup),(n_a,\leftarrow))$
$((m_a,\triangleright),(m_a,\rightarrow))$

$((n_a,a),(d_a,\sqcup))$
$((n_a,b),(t,\sqcup))$
$((n_a,c),(t,\sqcup))$
$((n_a,\sqcup),(t,\sqcup))$
$((n_a,\triangleright),(t,\rightarrow))$

$((d_a,a),(g_a,\leftarrow))$
$((d_a,b),(g_a,\leftarrow))$
$((d_a,c),(g_a,\leftarrow))$
$((d_a,\sqcup),(g_a,\leftarrow))$
$((d_a,\triangleright),(d_a,\rightarrow))$

$((g_a,a),(g_a,\leftarrow))$
$((g_a,b),(g_a,\leftarrow))$
$((g_a,c),(g_a,\leftarrow))$
$((g_a,\sqcup),(s,\rightarrow))$
$((g_a,\triangleright),(g_a,\rightarrow))$

$((q_b,a),(m_b,\rightarrow))$
$((q_b,b),(m_b,\rightarrow))$
$((q_b,c),(m_b,\rightarrow))$
$((q_b,\sqcup),(m_b,\rightarrow))$
$((q_b,\triangleright),(m_b,\rightarrow))$

$((m_b,a),(m_b,\rightarrow))$
$((m_b,b),(m_b,\rightarrow))$
$((m_b,c),(m_b,\rightarrow))$
$((m_b,\sqcup),(n_b,\leftarrow))$
$((m_b,\triangleright),(m_b,\rightarrow))$

$((n_b,a),(t,\sqcup))$
$((n_b,b),(d_b,\sqcup))$
$((n_b,c),(t,\sqcup))$
$((n_b,\sqcup),(t,\sqcup))$
$((n_b,\triangleright),(t,\rightarrow))$

$((d_b,a),(g_b,\leftarrow))$
$((d_b,b),(g_b,\leftarrow))$
$((d_b,c),(g_b,\leftarrow))$
$((d_b,\sqcup),(g_b,\leftarrow))$
$((d_b,\triangleright),(d_b,\rightarrow))$

$((g_b,a),(g_b,\leftarrow))$
$((g_b,b),(g_b,\leftarrow))$
$((g_b,c),(g_b,\leftarrow))$
$((g_b,\sqcup),(s,\rightarrow))$
$((g_b,\triangleright),(g_b,\rightarrow))$

$((t,a),(t,\leftarrow))$
$((t,b),(t,\leftarrow))$
$((t,c),(t,\leftarrow))$
$((t,\sqcup),(t,\leftarrow))$
$((t,\triangleright),(t,\rightarrow))$

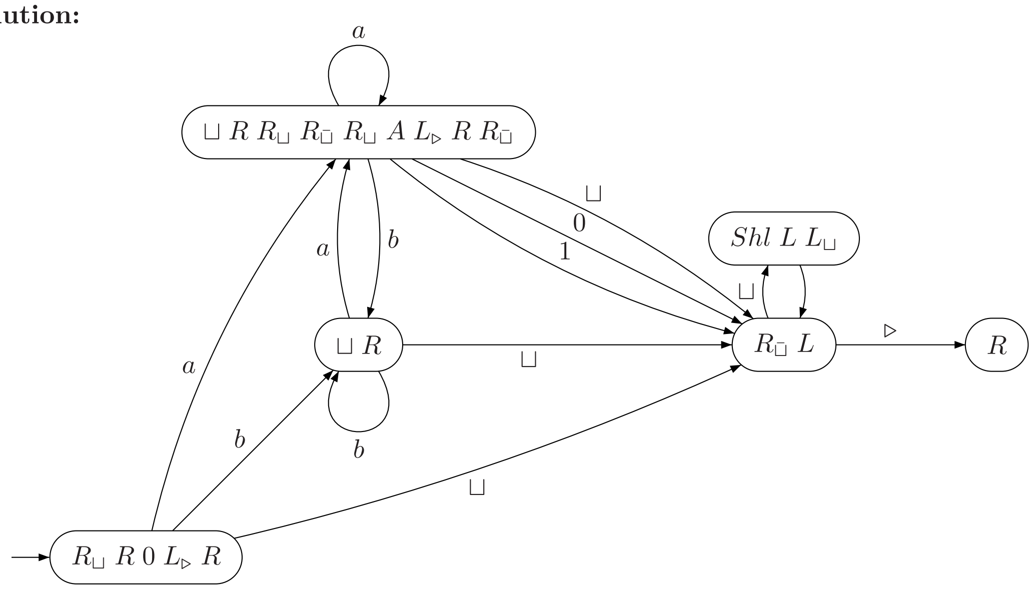
$((x,a),(t,a))$
$((x,b),(t,b))$
$((x,c),(t,c))$
$((x,\sqcup),(h,\sqcup))$
$((x,\triangleright),(t,\rightarrow))$

(b) Using the primitives R, L, $R_{\sqcup}$, $L_{\sqcup}$, $R_{\bar{\sqcup}}$, $L_{\bar{\sqcup}}$, $R_{\triangleright}$, $L_{\triangleright}$, $R_{\bar{\triangleright}}$, $L_{\bar{\triangleright}}$, R_0, L_0, R_1, L_1, $R_{\bar{0}}$, $L_{\bar{0}}$, $R_{\bar{1}}$, $L_{\bar{1}}$, R_a, L_a, $R_{\bar{a}}$, $L_{\bar{a}}$, R_b, L_b, $R_{\bar{b}}$, $L_{\bar{b}}$, Shl, Shr, A (add one), and S (subtract one) construct a Turing machine (in diagram form) that takes a word $w \in \{a, b\}^*$ and outputs the number of a's in binary form. For example, an input of $\triangleright\underline{b}bbbabbaaba$ produces $\triangleright\underline{1}00$.

The Turing machine A (add one) will add one to a number string given that the read/write head is on the space after the number and it returns the read/write head to the space after the number before it halts.

The Turing machine S (subtract one) will subtract one from a number string given that the read/write head is on the space after the number and it returns the read/write head to the space after the number before it halts.

Solution:



4. **Infinity:** (*10 Points*)**:** Prove that the cardinality of the power set of a set A, $\mathcal{P}(A)$, is strictly greater than the cardinality of A.

Solution: Since there is an injection $g : A \to \mathcal{P}(A)$ deffined by $g(a) = \{a\}$ we have $|A| \le |\mathcal{P}(A)|$. By way of contradiction assume that $|A| = |\mathcal{P}(A)|$, then there exists a bijection $f : A \to \mathcal{P}(A)$. Consider the set $A' = \{a \in A \mid a \notin f(a)\}$. Since $A' \in \mathcal{P}(A)$ there exists $a' \in A$ with $f(a') = A'$. Now either $a' \in A'$ or $a' \notin A'$. If $a' \in A'$ then by the deffinition of A', $a' \notin f(a') = A'$, a contradiction. If $a' \notin A'$ then again by the deffinition of A', $a' \in f(a') = A'$, a contradiction. Thus no bijection f exists and $|A| < |\mathcal{P}(A)|$.