Spring 2012 **Final Exam** Math 362

Final Exam

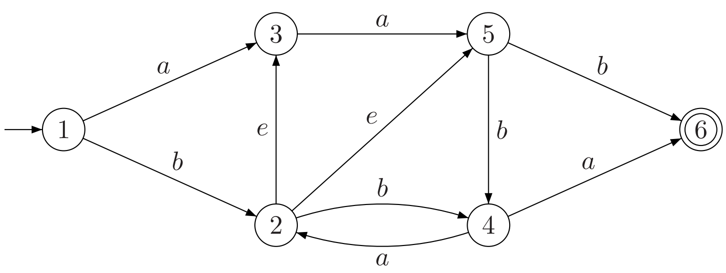
Name: ____

Write all of your responses on the extra exam paper provided. Turn in all work and this exam paper.

1. **Definitions and Short Answer:** (*4 Points Each*): Answer all of the following.
   (a) Define a Computation
   (b) Define a Regular Expression
   (c) What is the main computational difference between a finite automaton and a push-down automaton?
   (d) State the Church-Turing Thesis
   (e) Give definitions for the classes $\mathcal{P}$, $\mathcal{NP}$ and $\mathcal{NP}$-Complete.
   (f) Discuss what polynomial time reducible means.
   (g) Define a functional property.
   (h) State Cook's Theorem
   (i) State Rice's Theorem
   (j) State Post's Theorem

2. **True & False:** (*2 Points Each*) Mark each of the following as being either true or false.
   (a) ________ Any language accepted by a push-down automaton can be represented as a finite union of regular expressions.
   (b) ________ The union of two context-free languages is context-free.
   (c) ________ The Kleene star of a regular language is context-free.
   (d) ________ The intersection of a context-free language with a regular language is context-free.
   (e) ________ Gödel numbering is the process of putting the set of all Turing Machines in a one-to-one correspondence with the natural numbers.
   (f) ________ A nondeterministic Turing machine is more powerful than a deterministic Turing machine.
   (g) ________ The class of deterministic context-free languages is closed under complement.
   (h) ________ The first problem proven to be $\mathcal{NP}$-Complete was the Traveling Salesman Problem.
   (i) ________ There are a countable number of functions $f : \mathbb{N} \to \mathbb{N}$.
   (j) ________ It is possible for a set to have the same cardinality as a proper subset of itself.
   (k) ________ If it is ever proven that $\mathcal{P} = \mathcal{NP}$ then we will be able to convert all algorithms to equivalent algorithms that run in polynomial time.
   (l) ________ The certificate of a verification algorithm can be thought of as a solution to the problem.
   (m) ________ Decidable languages are closed under complement.
   (n) ________ A countable union of countable sets is countable.
   (o) ________ A programming language interpreter can be simulated by a Turing machine.
   (p) ________ Functional properties of programs are decidable.
   (q) ________ All algorithms can be polynomial bounded if we allow them to be solved by nondeterministic Turing machines.
   (r) ________ Polynomial reducibility is an equivalence relation.
   (s) ________ Every algorithm in $\mathcal{NP}$ is polynomial reducible to the Traveling Salesman Problem.
   (t) ________ The main purpose of Chomsky Normal Form is to solve the membership problem.

3. **Finite Automata:** (*20 Points*): Consider the following nondeterministic finite automaton.



   (a) Is *bbaaa* acceptable or not acceptable? If it is acceptable display a sequence of states for the word that end in a favorable state.
   (b) Is *bbababa* acceptable or not acceptable? If it is acceptable display a sequence of states for the word that end in a favorable state.
   (c) Convert the automaton to a regular expression.

4. **Context-Free Languages and Grammars:** (*20 Points*): Consider the following context-free grammar.

$$G = \{S \to AB, A \to bAa, B \to aBb, A \to e, A \to b, B \to e, B \to b\}$$

   (a) Give a derivation for *bbaaaabbb*.
   (b) Display the parse tree for the above derivation.
   (c) Describe the language $L(G)$.
   (d) Is $L(G)$ regular? Why or why not?
   (e) Convert the grammar to Chomsky Normal Form.

5. **Push-Down Automata:** (*20 Points*) Construct a PDA, A, that accepts the language $L = \{w \mid n_a(w) = n_b(w)\}$. Note that in this exercise $\Sigma = \{a, b\}$ and $n_a(w)$ stands for the number of a's in w and $n_b(w)$ stands for the number of b's in w. Then use a transition/state/input/stack table (as done in class) to test the strings *aababb* and *bbabaaba*.

6. **Turning Machines:** (*20 Points Each*): Answer all of the following.

   (a) Give a complete set of transitions for the Turing machine that will take as input a binary number and decide if the number is divisible by 4. For example, an input of $\triangleright\underline{1}001001$ will produce an output of $\triangleright\underline{0}$ and an input of $\triangleright\underline{1}1100100$ will produce an output of $\triangleright\underline{1}$. Hint: write the binary representations of 4, 8, 12, ... and notice what they all have in common.
   (b) Using primitives construct a Turing machine (in diagram form) that semidecides the language $L = \{w \mid n_a(w) = n_b(w)\}$.

7. **Proofs:** (*10 Points*): Do one and only one of the following.

   (a) Prove that the language $L = \{a^{k^2}, k = 0, 1, 2, \ldots\}$ is not regular.
   (b) Prove that the language $L = \{a^k b^k c^k | k = 0, 1, 2, \ldots\}$ is not context-free.
   (c) Show that the language $L = \{\langle P\rangle | P$ is equivalent to a given program $P_0\}$ is undecidable.

8. **Proofs:** (*10 Points*): Do one and only one of the following.

   (a) Given that languages L and M are acceptable by finite automata show that $L \cup M$, LM and L^* are acceptable by finite automata.
   (b) Given that L and M are two context-free languages show that $L \cup M$, LM and L^* are context-free languages.
   (c) Let $H = \{\langle M\rangle\langle w\rangle \mid M$ halts on $w\}$ and show that this language is not decidable.
   (d) Prove that $H_{\mathcal{P}} = \{\langle M\rangle\langle w\rangle \mid M$ accepts w in at most $2^{|w|}$ steps $\}$ is not an element of $\mathcal{P}$.

9. **Proofs:** (*10 Points*): Do one and only one of the following.

   (a) Prove that the cardinality of the set of all Turing machines is countable.
   (b) Prove that the cardinality of the power set of a set A is strictly greater than the cardinality of A.
   (c) Prove that there exists a function $f : \mathbb{N} \to \mathbb{N}$ that is not partial Turing computable.