Fall 2013

Exam #1 Key

COSC 362

1. (*10 Points*) Do one and only one of the following proofs,

   (a) Prove by induction that for $n \geq 4$, $2^n < n!$.

   Solution: If $n = 4$, then $2^n = 2^4 = 16 < 24 = 4! = n!$. Assume that the result is true for n (and that $n > 4$), then

   $$2^{n+1} = 2^n \cdot 2 < n! \cdot 2 < n! \cdot (n+1) = (n+1)!$$

   (b) Prove by induction that for $\displaystyle\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$.

   Solution: Let $n = 1$, then $\sum_{i=1}^{1} \frac{1}{i^2} = 1 \leq 2 - \frac{1}{1}$. Assume that the result is true for n (and that $n > 1$), then

   $$\begin{aligned}
   \sum_{i=1}^{n+1} \frac{1}{i^2} &= \frac{1}{(n+1)^2} + \sum_{i=1}^{n} \frac{1}{i^2} \\
   &\leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \\
   &= 2 - \left(\frac{1}{n} - \frac{1}{(n+1)^2}\right) \\
   &= 2 - \left(\frac{(n+1)^2 - n}{n(n+1)^2}\right) \\
   &= 2 - \left(\frac{n^2 + n + 1}{n(n+1)^2}\right) \\
   &< 2 - \left(\frac{n^2 + n}{n(n+1)^2}\right) \\
   &= 2 - \left(\frac{n(n+1)}{n(n+1)^2}\right) \\
   &= 2 - \frac{1}{n+1}
   \end{aligned}$$

2. (*5 Points Each*) Answer each of the following questions on languages and grammars. For this exercise, $\Sigma = \{a, b\}$.

   (a) Give a grammar for the language L_1 of all odd-length palindromes.
   Solution:

$$\begin{aligned} S &\longrightarrow aSa \\ S &\longrightarrow bSb \\ S &\longrightarrow a \\ S &\longrightarrow b \end{aligned}$$

   (b) Give a grammar for the language $L_2 = \{w \in \Sigma^* \mid n_a(w) = n_b(w) + 1\}$.
   Solution:

$$\begin{aligned} S &\longrightarrow AaA \\ A &\longrightarrow AaAbA \\ A &\longrightarrow AbAaA \\ A &\longrightarrow \lambda \end{aligned}$$

   (c) Give a grammar for the language $L_1 \cup L_2$.
   Solution: $S \longrightarrow S_1 \mid S_2$, and

$$\begin{aligned} S_1 &\longrightarrow aS_1a \\ S_1 &\longrightarrow bS_1b \\ S_1 &\longrightarrow a \\ S_1 &\longrightarrow b \\ S_2 &\longrightarrow AaA \\ A &\longrightarrow AaAbA \\ A &\longrightarrow AbAaA \\ A &\longrightarrow \lambda \end{aligned}$$

   (d) Give a grammar for the language L_1L_2.
   Solution: $S \longrightarrow S_1S_2$, and

$$\begin{aligned} S_1 &\longrightarrow aS_1a \\ S_1 &\longrightarrow bS_1b \\ S_1 &\longrightarrow a \\ S_1 &\longrightarrow b \\ S_2 &\longrightarrow AaA \\ A &\longrightarrow AaAbA \\ A &\longrightarrow AbAaA \\ A &\longrightarrow \lambda \end{aligned}$$

3. (*5 Points Each*) For each of the following languages, give a regular expression for that language. For this exercise, $\Sigma = \{a, b\}$.

   (a) $L = \{a^n b^m \mid (n+m) \text{ is even}\}$

   Solution: $(aa)^*(bb)^* + (aa)^*ab(bb)^*$

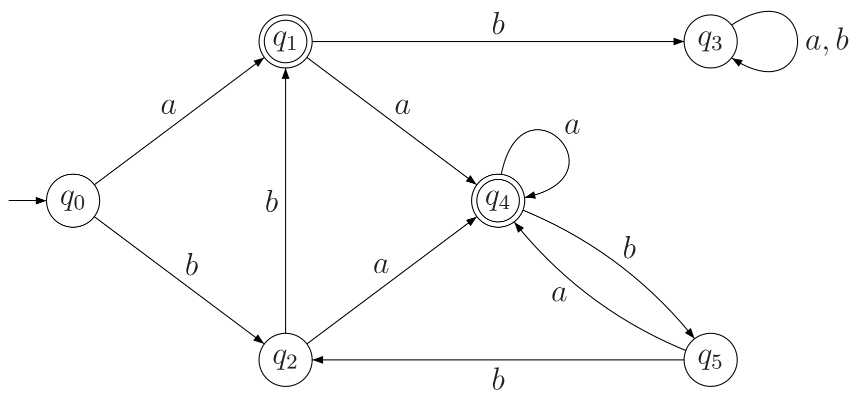
   (b) L is the language of all words with at most two occurrences of the substring aa.

   Solution: $(b+ab)^*aab(b+ab)^*aa(b+ba)^* + (b+ab)^*aaa(b+ba)^* + (b+ab)^*aa(b+ba)^* + (b+ab)^*(a+\lambda)$

   (c) $L = \{vwv \mid w \in \Sigma^* \text{ and } 1 \leq |v| \leq 2\}$

   Solution: $a(a+b)^*a + b(a+b)^*b + aa(a+b)^*aa + ab(a+b)^*ab + ba(a+b)^*ba + bb(a+b)^*bb$

4. (*30 Points*) Consider the following DFA, A.



   (a) Determine if the automaton accepts the following words. Display the sequence of states for each word.

      i. $aabbaa$ — q_0, q_1, q_4, q_5, q_2, q_4, q_4 — Accepted.

      ii. $bbaabbab$ — q_0, q_2, q_1, q_4, q_4, q_5, q_2, q_4, q_5 — Not Accepted.

      iii. $aababab$ — q_0, q_1, q_4, q_5, q_4, q_5, q_4, q_5 — Not Accepted.

   (b) Is $L(aa(ba)^*) \subseteq L(A)$? Why or why not?

   Solution: Yes, aa drives the automaton to q_4, from there any number of sequences of ba take you back to q_4. Hence you always end in the final state q_4 so the words are accepted.

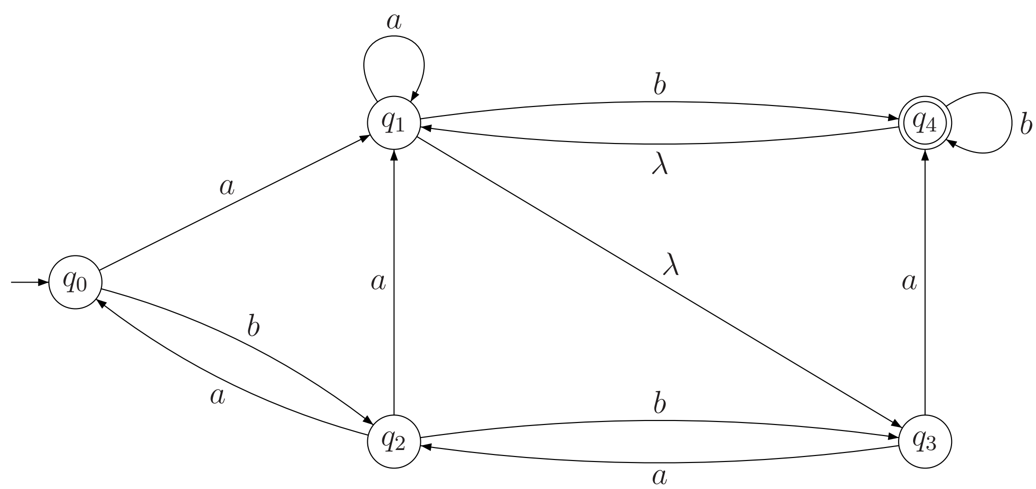
   (c) For what values of n and m is $a^n b^m \in L(A)$?

   Solution: $n = 1$ and $m = 0$, or $n \geq 2$ and either $m = 0$ or $m = 3$.

   (d) What is the smallest run of b's that will guarantee that the word will not be accepted. That is, if $w = ub^n v$ for any $u, v \in \Sigma^*$, what is the smallest value of n will guarantee that $w \notin L(A)$? Justify your answer.

   Solution: 4, with $w = ub^n v$ for $v \in \Sigma^*$, the only way to guarantee that the word will not be accepted is to end in the dead state q_3. From each state, a run of 1–4 b's will drive the automaton to q_3.

5. (*35 Points*) Consider the following NFA, A.



(a) Determine if the automaton accepts the following words. If it does, display the sequence of states that drive the word to a final state.

i. $abab$ — q_0, q_1, q_4, q_1, q_1, q_4 — Accepted.

ii. $abbbaaba$ — q_0, q_1, q_4, q_4, q_4, q_1, q_1, q_1, q_4, q_1, q_3, q_4 — Accepted.

iii. $bbaab$ — q_0, q_2, q_3, q_4, q_1, q_3, q_4, q_4 — Accepted.

(b) Find a word of length 4 that is not accepted.

Solution: $bbbb$ and $bbba$.

(c) Describe the language that is accepted by this automaton, $L(A)$.

Solution: $(aa + ab + baa + bab + bba)(a + b)^*$

(d) Convert this NFA to a DFA.