Exam #3 Key

1. (*20 Points*) Construct a Context-Free Grammar for the language

$$L = \{a^n b^m c^k \mid k = |n - m|, n \geq 0, m \geq 0\}$$

Solution: Since $k = |n - m|$, this means that either $k = n - m$ or $k = m - n$, that is, either $n = m + k$ or $m = k + n$. The first can be constructed by

$$
\begin{aligned}
S &\rightarrow aSc \mid A \mid \lambda \\
A &\rightarrow aAb \mid \lambda
\end{aligned}
$$

and the second

$$
\begin{aligned}
S &\rightarrow BC \mid \lambda \\
B &\rightarrow aBb \mid \lambda \\
C &\rightarrow bCc \mid \lambda
\end{aligned}
$$

putting these altogether,

$$
\begin{aligned}
S &\rightarrow S_1 \mid S_2 \\
S_1 &\rightarrow aS_1c \mid A \mid \lambda \\
A &\rightarrow aAb \mid \lambda \\
S_2 &\rightarrow BC \mid \lambda \\
B &\rightarrow aBb \mid \lambda \\
C &\rightarrow bCc \mid \lambda
\end{aligned}
$$

2. (*15 Points*) Show that the following grammar is ambiguous.

$$
\begin{aligned}
S &\rightarrow abAAB \mid BaCbB \mid \lambda \\
A &\rightarrow aaA \mid AbbAC \mid b \\
B &\rightarrow AaA \mid bb \\
C &\rightarrow CAB \mid bB \mid \lambda
\end{aligned}
$$

Solution: Consider the following two distinct left-most derivations of *bbabbbbb*,

$S \rightarrow BaCbB \rightarrow bbaCbB \rightarrow bbaCABbB \rightarrow bbaABbB \rightarrow bbabBbB \rightarrow bbabbbbB \rightarrow bbabbbbb$

$S \rightarrow BaCbB \rightarrow bbaCbB \rightarrow bbabBbB \rightarrow bbabbbbB \rightarrow bbabbbbb$

3. (*25 Points*) Convert the following grammar to Chomsky Normal Form.

$$
\begin{aligned}
S &\rightarrow AAB \,|\, BaCbD \,|\, B \\
A &\rightarrow aA \,|\, bbAC \,|\, b \\
B &\rightarrow AaA \,|\, b \,|\, a \,|\, \lambda \\
C &\rightarrow AB \,|\, bB \\
D &\rightarrow CaB \,|\, a
\end{aligned}
$$

Solution: First remove all λ-productions, $V_N = \{S, B\}$

$$
\begin{aligned}
S &\rightarrow AAB \,|\, AA \,|\, BaCbD \,|\, aCbD \,|\, B \\
A &\rightarrow aA \,|\, bbAC \,|\, b \\
B &\rightarrow AaA \,|\, b \,|\, a \\
C &\rightarrow AB \,|\, A \,|\, bB \,|\, b \\
D &\rightarrow CaB \,|\, Ca \,|\, a
\end{aligned}
$$

Now remove all of the unit productions,

$$
\begin{aligned}
S &\rightarrow AAB \,|\, AA \,|\, BaCbD \,|\, aCbD \,|\, AaA \,|\, b \,|\, a \\
A &\rightarrow aA \,|\, bbAC \,|\, b \\
B &\rightarrow AaA \,|\, b \,|\, a \\
C &\rightarrow AB \,|\, aA \,|\, bbAC \,|\, bB \,|\, b \\
D &\rightarrow CaB \,|\, Ca \,|\, a
\end{aligned}
$$

There are no useless productions in this grammar. The intermediate step is,

$$
\begin{aligned}
S &\rightarrow AAB \,|\, AA \,|\, BX_aCX_bD \,|\, X_aCX_bD \,|\, AX_aA \,|\, b \,|\, a \\
A &\rightarrow X_aA \,|\, X_bX_bAC \,|\, b \\
B &\rightarrow AX_aA \,|\, b \,|\, a \\
C &\rightarrow AB \,|\, X_aA \,|\, X_bX_bAC \,|\, X_bB \,|\, b \\
D &\rightarrow CX_aB \,|\, CX_a \,|\, a \\
X_a &\rightarrow a \\
X_b &\rightarrow b
\end{aligned}
$$

Finally,

$$\begin{aligned}
S &\rightarrow AD_1 \mid AA \mid BD_2 \mid X_aCX_bD \mid AX_aA \mid b \mid a \\
D_1 &\rightarrow AB \\
D_2 &\rightarrow X_aD_3 \\
D_3 &\rightarrow CD_4 \\
D_4 &\rightarrow X_bD \\
A &\rightarrow X_aA \mid X_bD_5 \mid b \\
D_5 &\rightarrow X_bD_6 \\
D_6 &\rightarrow AC \\
B &\rightarrow AD_7 \mid b \mid a \\
D_7 &\rightarrow X_aA \\
C &\rightarrow AB \mid X_aA \mid X_bD_8 \mid X_bB \mid b \\
D_8 &\rightarrow X_bD_6 \\
D &\rightarrow CD_9 \mid CX_a \mid a \\
D_9 &\rightarrow X_aB \\
X_a &\rightarrow a \\
X_b &\rightarrow b
\end{aligned}$$

4. (*20 Points*) Construct a Nondeterministic Push-Down Automaton for the following language. Use it to test the words *abaabaaa* and *ababaa* for acceptance.

$$L = \{w \mid n_a(w) = 3n_b(w)\}$$

Solution:

$$\begin{aligned}
\delta(q_0, a, z) &= \{(q_0, 0z)\} \\
\delta(q_0, a, 1) &= \{(q_0, \lambda)\} \\
\delta(q_0, b, z) &= \{(q_0, 111z)\} \\
\delta(q_0, b, 1) &= \{(q_0, 1111)\} \\
\delta(q_0, b, 0) &= \{(q_1, \lambda)\} \\
\delta(q_1, \lambda, 0) &= \{(q_2, \lambda)\} \\
\delta(q_2, \lambda, 0) &= \{(q_0, \lambda)\} \\
\delta(q_1, \lambda, z) &= \{(q_0, 11z)\} \\
\delta(q_2, \lambda, z) &= \{(q_0, 1z)\} \\
\delta(q_0, \lambda, z) &= \{(q_f, z)\}
\end{aligned}$$

3

5. (*20 Points*) Convert the following Context-Free Grammar into a Nondeterministic Push-Down Automaton.

$$
\begin{aligned}
S &\rightarrow AAB \mid BaCbD \mid B \\
A &\rightarrow aA \mid bbAC \mid b \\
B &\rightarrow AaA \mid b \mid a \\
C &\rightarrow AB \mid bB \\
D &\rightarrow CaB \mid a
\end{aligned}
$$

Solution: First we do a replacement of some terminals with variables, where needed,

$$
\begin{aligned}
S &\rightarrow AAB \mid BX_aCX_bD \mid B \\
A &\rightarrow aA \mid bX_bAC \mid b \\
B &\rightarrow AX_aA \mid b \mid a \\
C &\rightarrow AB \mid bB \\
D &\rightarrow CX_aB \mid a \\
X_a &\rightarrow a \\
X_b &\rightarrow b
\end{aligned}
$$

Now the conversion step,

$$
\begin{aligned}
\delta(q_0, a, z) &= \{(q_0, 0z)\} \\
\delta(q_0, a, 1) &= \{(q_0, \lambda)\} \\
\delta(q_0, b, z) &= \{(q_0, 111z)\} \\
\delta(q_0, b, 1) &= \{(q_0, 1111)\} \\
\delta(q_0, b, 0) &= \{(q_1, \lambda)\} \\
\delta(q_1, \lambda, 0) &= \{(q_2, \lambda)\} \\
\delta(q_2, \lambda, 0) &= \{(q_0, \lambda)\} \\
\delta(q_1, \lambda, z) &= \{(q_0, 11z)\} \\
\delta(q_2, \lambda, z) &= \{(q_0, 1z)\} \\
\delta(q_0, \lambda, z) &= \{(q_f, z)\}
\end{aligned}
$$

6. (*10 Points*) Show that every regular language is a deterministic context-free language.

Solution: