1. (10 points) Use induction to prove that for all $n \geq 1$,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

**Solution:** When $n = 1$ we have

$$\sum_{i=1}^{1} i^2 = 1 = \frac{1(1+1)(2+1)}{6}$$

Assume that the statement is true for all $k \leq n$ and prove the statement for the case of $n + 1$,

$$\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^{n} i^2 + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\
&= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)(n+2)(2(n+1)+1)}{6}
\end{aligned}$$

2. (25 Points) Answer each of the following questions on languages and grammars. For this exercise, $\Sigma = \{a, b\}$.

   (a) Find a grammar that generates the language of all palindromes, that is,

$$L = \{w \in \Sigma^* \mid w = w^R\}$$

   **Solution:**

$$\begin{aligned}
S &\rightarrow aSa \\
S &\rightarrow bSb \\
S &\rightarrow a \\
S &\rightarrow b \\
S &\rightarrow \lambda
\end{aligned}$$

   (b) Find a grammar that generates the language,

$$L = \{a^n b^m \mid n \geq 0 \text{ and } m \geq 0\}$$

**Solution:**

$$\begin{array}{rcl} S & \rightarrow & AB \\ A & \rightarrow & aA \\ B & \rightarrow & bB \\ A & \rightarrow & \lambda \\ B & \rightarrow & \lambda \end{array}$$

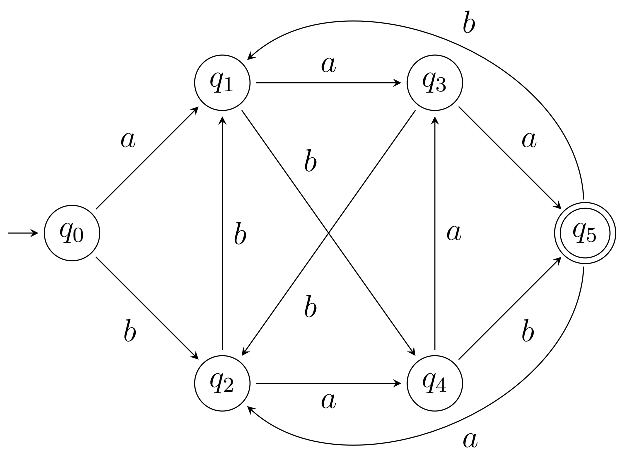
(c) Find a grammar that generates the union of the above two languages, specifically,

$$L = \{w \in \Sigma^* \mid w = w^R\} \cup \{a^n b^m \mid n \geq 0 \text{ and } m \geq 0\}$$

**Solution:**

$$\begin{array}{rcl} S & \rightarrow & S_1 \\ S_1 & \rightarrow & aS_1a \\ S_1 & \rightarrow & bS_1b \\ S_1 & \rightarrow & a \\ S_1 & \rightarrow & b \\ S_1 & \rightarrow & \lambda \end{array} \qquad \begin{array}{rcl} S & \rightarrow & S_2 \\ S_2 & \rightarrow & AB \\ A & \rightarrow & aA \\ B & \rightarrow & bB \\ A & \rightarrow & \lambda \\ B & \rightarrow & \lambda \end{array}$$

3. (20 Points) Consider the following DFA, $M$




(a) Determine if the automaton accepts the following words. Display the sequence of states for each word.

i. $abbaba$
**Solution:** $abbaba$ — $q_0, q_1, q_4, q_5, q_2, q_1, q_3$, Not Accepted

ii. $bbbbbbb$
**Solution:** $bbbbbbb$ — $q_0, q_2, q_1, q_4, q_5, q_1, q_4, q_5$, Accepted

iii. $ababab$
**Solution:** $ababab$ — $q_0, q_1, q_4, q_3, q_2, q_4, q_5$, Accepted

iv. $bbbabbb$
**Solution:** $bbbabbb$ — $q_0, q_2, q_1, q_4, q_3, q_2, q_1, q_4$, Not Accepted

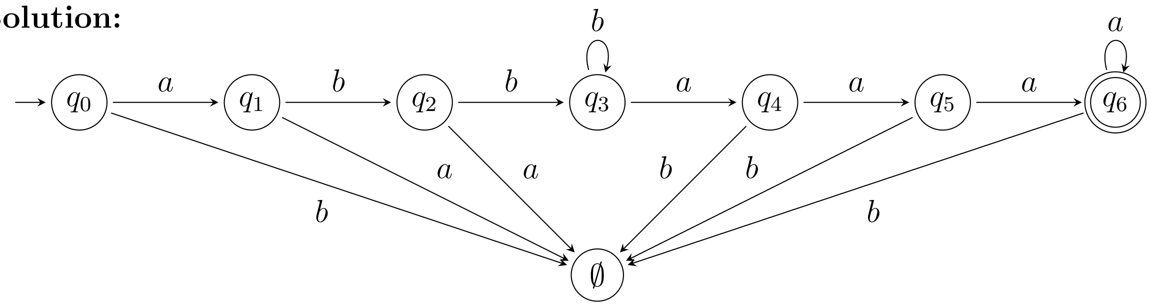
(b) If $a^n \in L(M)$ then what are all of the possible values of $n$.
**Solution:** $n = 3 + 4k$ where $k \geq 1$.

(c) If $b^n \in L(M)$ then what are all of the possible values of $n$.
**Solution:** $n = 4 + 3k$ where $k \geq 1$.

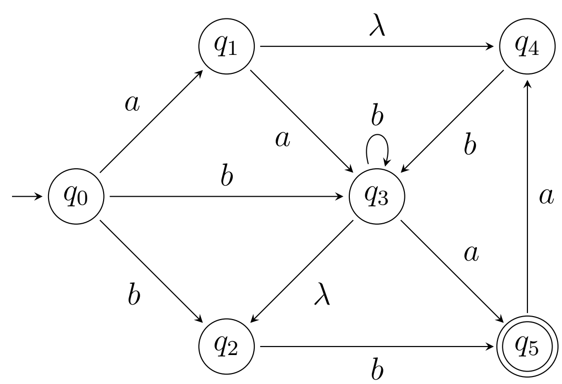
4. (10 Points) Construct a DFA, with $\Sigma = \{a, b\}$, that accepts the language

$$L = \{ab^n a^m \mid n \geq 2 \text{ and } m \geq 3\}$$

**Solution:**




5. (25 Points) Consider the following NFA, $M$




(a) Determine if the automaton accepts the following words. Display the sequence of states for each accepted word and if the word is not accepted, give a short explanation of why.

i. *abbaba*
**Solution:** *abbaba* — $q_0, q_1, q_4, q_3, q_2, q_5, q_4, q_3, q_5$, Accepted

ii. *bbbbbbb*
**Solution:** *bbbbbbb* — $q_0, q_3, q_3, q_3, q_3, q_3, q_3, q_2, q_5$, Accepted

iii. *ababab*
**Solution:** *ababab* — Not Accepted: The *aba* path ends on $q_5$ and where there is no way to read a $b$.

iv. *bbbabbb*
**Solution:** *bbbabbb* — $q_0, q_3, q_3, q_2, q_5, q_4, q_3, q_3, q_2, q_5$, Accepted

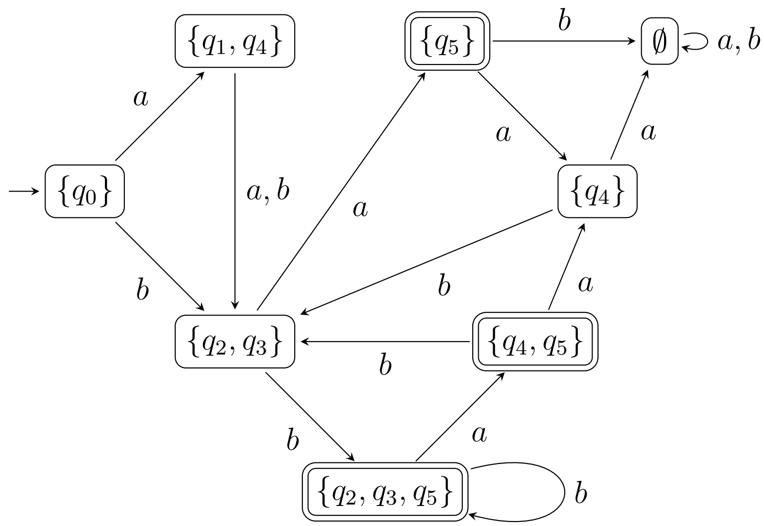
(b) If $a^n \in L(M)$ then what are all of the possible values of $n$.
**Solution:** $n = 3$

(c) If $b^n \in L(M)$ then what are all of the possible values of $n$.

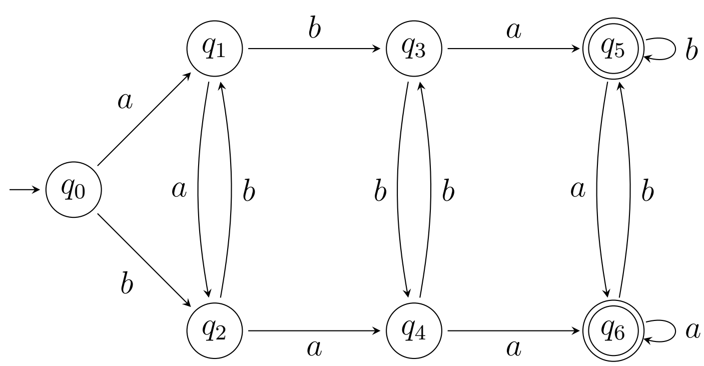
**Solution:** $n \geq 2$

(d) Convert the NFA to a DFA

**Solution:**



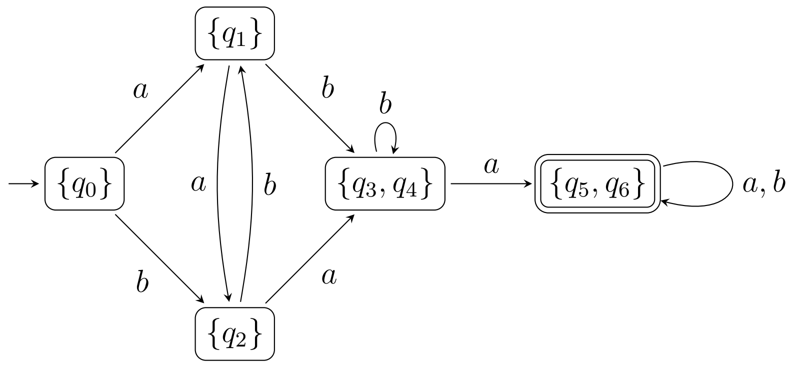

6. (20 Points) Minimize the number of states of the following DFA, and display the minimal state automaton. Make sure you show all steps in the conversion algorithm.




**Solution:** The distinguishable state chart is below,

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $D$ | 01 | $\to$ | 12, 23 | $D$ | 13 | $\to$ | 25, 34 | $D$ | 26 | | |
| $D$ | 02 | $\to$ | 14, 21 | $D$ | 14 | $\to$ | 26, 33 | | 34 | $\to$ | 56, 43 |
| $D$ | 03 | $\to$ | 15, 24 | $D$ | 15 | | | $D$ | 35 | | |
| $D$ | 04 | $\to$ | 16, 23 | $D$ | 16 | | | $D$ | 36 | | |
| $D$ | 05 | | | $D$ | 23 | $\to$ | 45, 14 | $D$ | 45 | | |
| $D$ | 06 | | | $D$ | 24 | $\to$ | 46, 13 | $D$ | 46 | | |
| $D$ | 12 | $\to$ | 24, 31 | $D$ | 25 | | | | 56 | $\to$ | 66, 55 |

Hence the equivalence classes, and thus the new nodes are, $\{q_0\}$, $\{q_1\}$, $\{q_2\}$, $\{q_3, q_4\}$, $\{q_5, q_6\}$. So the equivalent automaton with minimal states is,


$\{q_1\}$
$\{q_0\}$
$\{q_3, q_4\}$
$\{q_5, q_6\}$
$\{q_2\}$
$a$
$b$
$a, b$