1. (5 points Each) Find regular expressions for the following languages, $\Sigma = \{a, b\}$.

   (a) $L = \{a^m b^n w \mid w \in \Sigma^*, m \geq 4, n \leq 3\}$
   Solution: $aaaaa^*(\lambda + b + bb + bbb)(a + b)^*$

   (b) $L = \{vwv \mid w \in \Sigma^*, |v| = 3\}$
   Solution: $aaa(a+b)^*aaa + aab(a+b)^*aab + aba(a+b)^*aba + baa(a+b)^*baa + abb(a+b)^*abb + bab(a+b)^*bab + bba(a+b)^*bba + bbb(a+b)^*bbb$

   (c) $L = \{w \mid n_a(w) \mod 3 = 1\}$
   Solution: $b^*a(b^*ab^*ab^*ab^*)^*b^*$

   (d) $L = \{w \mid w \in \Sigma^*, w$ contains exactly one pair of consecutive a's$\}$
   Solution: $b^*(abb^*)^*aa(bb^*a)^*b^*$

2. (10 points Each) Prove the following,

   (a) Given a set of n regular languages $\{L_1, L_2, L_3, \ldots, L_n\}$, show that the union of these is a regular language, that is, show that $L = L_1 \cup L_2 \cup L_3 \cup \cdots \cup L_n$ is regular.
   Solution: Using induction, if $n = 2$, we know that the union of two regular languages is regular, so $L = L_1 \cup L_2$ is regular. Assume that the theorem holds for a set of k regular languages and we will show the $k + 1$ case. Let

$$\begin{aligned} L &= L_1 \cup L_2 \cup L_3 \cup \cdots \cup L_k \cup L_{k+1} \\ &= (L_1 \cup L_2 \cup L_3 \cup \cdots \cup L_k) \cup L_{k+1} \\ &= M \cup L_{k+1} \quad (M \text{ is regular by the inductive hypothesis.}) \end{aligned}$$

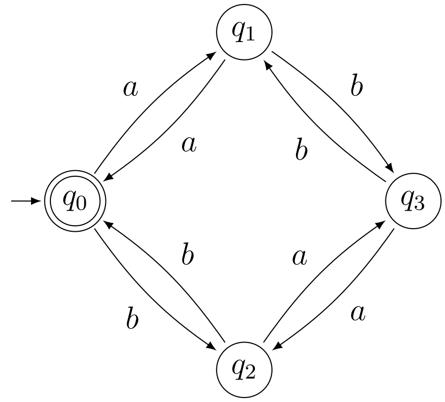
   So L is the union of two regular languages and is hence regular.

   (b) Given two regular languages L_1 and L_2 show that the reversal difference, R, is regular. The reversal difference is defined to be

$$R = \{w \in \Sigma^* \mid w \in L_1 \text{ and } w^R \notin L_2\}$$

   Solution: $R = L_1 \cap \overline{L_2^R}$, since the reversal and complement of a regular language is regular and the intersection of two regular languages is regular, R is regular.

3. (20 points) The following finite automaton is one possible automaton for the language $L = \{w \in \{a, b\}^* \mid n_a(w)$ and $n_b(w)$ are both even.$\}$.



(a) Using the algorithm discussed in class, convert this automaton to a regular grammar.

Solution: We will take the starting symbol as q_0.

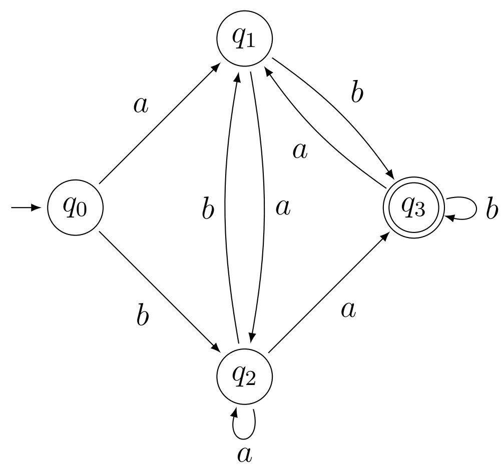
$$\begin{array}{lcl@{\qquad}lcl} q_0 & \to & aq_1 & q_2 & \to & aq_3 \\ q_0 & \to & bq_2 & q_2 & \to & bq_0 \\ q_1 & \to & aq_0 & q_3 & \to & aq_2 \\ q_1 & \to & bq_3 & q_3 & \to & bq_1 \\ & & & q_0 & \to & \lambda \end{array}$$

(b) Use the grammar to derive the word $aababa$.

Solution:

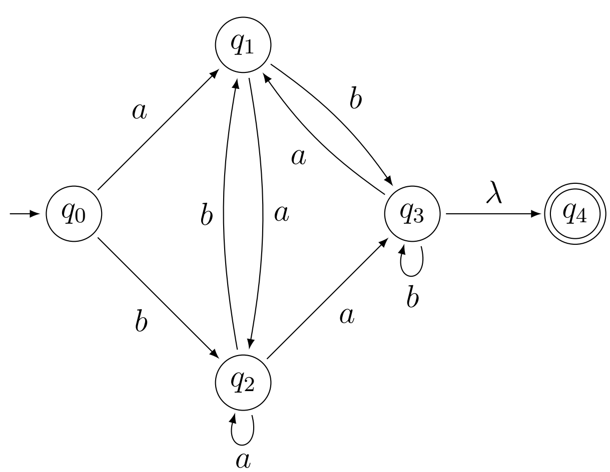
$$q_0 \to aq_1 \to aaq_0 \to aabq_2 \to aabaq_3 \to aababq_1 \to aababaq_0 \to aababa$$

4. (20 points) Consider the following automaton. Using the algorithm discussed in class, convert this automaton to a regular expression. Show all of your steps in the conversion process.

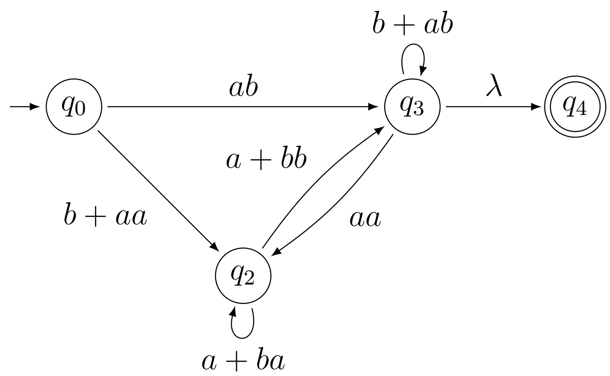


Solution: $(ab + (b + aa)(a + ba)^*(a + bb))(b + ab + aa(a + ba)^*(a + bb))^*$

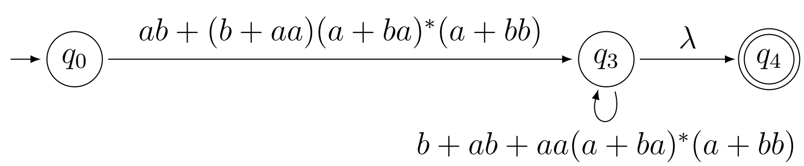
Step 1: Convert



Step 2: Remove q_1



Step 3: Remove q_2



Step 4: Remove q_3 to obtain the expression

$$(ab + (b + aa)(a + ba)^*(a + bb))(b + ab + aa(a + ba)^*(a + bb))^*$$

5. (10 Points Each) For each of the following languages, determine if it is regular or not regular, justify your answers.

   (a) $L = \{a^n b^q a^k \mid n = q \text{ or } k \neq q\}$

   Solution: L is not regular.

   Assume that L is regular, then there exists a number $m > 0$ such that for any word $w \in L$ with $|w| \geq m$, we can write $w = xyz$ with $|xy| \leq m$, $|y| \geq 1$, and $xy^i z \in L$ for all $i = 0, 1, 2, 3, \ldots$. Let $w = a^m b^m a^m$, since $w \in L$ and $|w| = 3m \geq m$ we can write $w = xyz$ with the above restrictions. Hence, $y = a^k$ for some $1 \leq k \leq m$, and $xy^2 z = a^{m+k} b^m a^m \notin L$ since $n \neq q$ and $k = q$. This contradiction proves that L is not regular.

   (b) $L = \{a^n b^k \mid n < 2k\}$

   Solution: L is not regular.

   Assume that L is regular, then there exists a number $m > 0$ such that for any word $w \in L$ with $|w| \geq m$, we can write $w = xyz$ with $|xy| \leq m$, $|y| \geq 1$, and $xy^i z \in L$ for all $i = 0, 1, 2, 3, \ldots$. Let $w = a^{2m-1} b^m$, since $w \in L$ and $|w| = 3m - 1 \geq m$ we can write $w = xyz$ with the above restrictions. Hence, $y = a^k$ for some $1 \leq k \leq m$, and $xy^2 z = a^{2m-1+k} b^m \notin L$ since $2m - 1 + k \geq 2m - 1 + 1 = 2m$. This contradiction proves that L is not regular.

6. (10 Points) Prove or disprove the following statement: If L_1 and L_2 are nonregular languages, then $L_1 \cup L_2$ is also a nonregular language.

   Solution: False: Let L_1 be any nonregular language and let $L_2 = \overline{L_1}$. L_2 is not regular, since if it was then $\overline{L_2} = \overline{\overline{L_1}} = L_1$ would be regular, which is contrary to our assumption. So both L_1 and L_2 are nonregular languages, but $L_1 \cup L_2 = \Sigma^* = (a+b)^*$ is regular.