1. (*10 Points Each*) Construct context-free grammars for the following languages,

   (a) $L = \{a^n b^{n+m} c^m \mid n \geq 0 \text{ and } m \geq 0\}$
   Solution:

$$\begin{aligned} S &\rightarrow AB \\ A &\rightarrow aAb|\lambda \\ B &\rightarrow bBc|\lambda \end{aligned}$$

   (b) $L = \{wcw^R \mid w \in \{a,b\}^*\}$
   Solution:

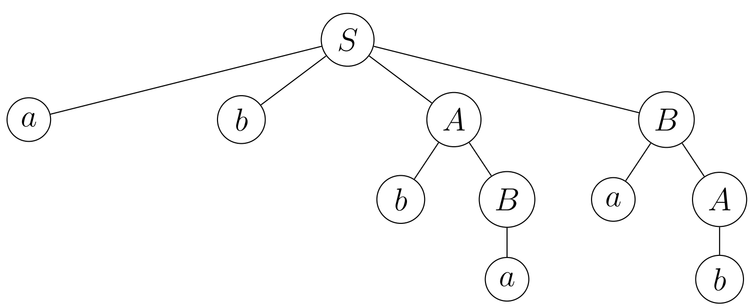
$$S \rightarrow aSa|bSb|c$$

2. (*5 Points Each*) Consider the following context free grammar, G,

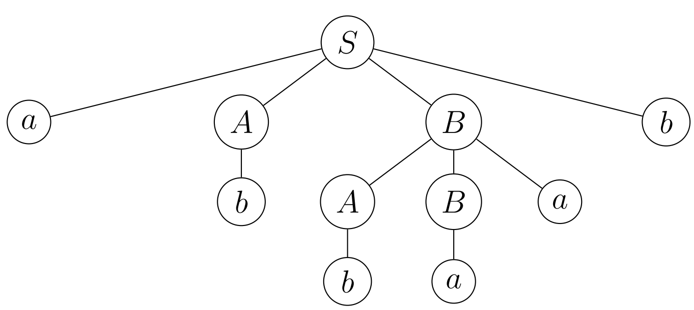
$$\begin{aligned} S &\rightarrow aABb|aBAb|abAB \\ A &\rightarrow aBA|BAb|bB|b \\ B &\rightarrow bAB|ABa|aA|a \end{aligned}$$

   (a) Find a leftmost derivation for the word $abbaab$.
   Solution: $S \rightarrow abAB \rightarrow abbBB \rightarrow abbaB \rightarrow abbaaA \rightarrow abbaab$

   (b) Construct the derivation tree for the derivation in 2a.
   Solution:



   (c) Show that this grammar is ambiguous.
   Solution: $S \rightarrow aABb \rightarrow abBb \rightarrow abABab \rightarrow abbBab \rightarrow abbaab$



3. (*10 Points Each*) Create nondeterministic push-down automata that accept the following languages,

   (a) $L = L(ba^*abb) \cup L(a^*ba^*)$

   Solution: q_0 will be the starting state, z will represent the bottom of the stack, and q_f will be the only final state.

$$\delta(q_0, \lambda, z) = \{(q_1, z), (q_5, z)\}$$

$$\begin{array}{rclrcl} \delta(q_1, b, z) &=& \{(q_2, z)\} & \delta(q_5, a, z) &=& \{(q_5, z)\} \\ \delta(q_2, a, z) &=& \{(q_3, z)\} & \delta(q_5, b, z) &=& \{(q_6, z)\} \\ \delta(q_3, a, z) &=& \{(q_3, z)\} & \delta(q_6, a, z) &=& \{(q_6, z)\} \\ \delta(q_3, b, z) &=& \{(q_4, z)\} & \delta(q_6, \lambda, z) &=& \{(q_f, z)\} \\ \delta(q_4, b, z) &=& \{(q_f, z)\} & & & \end{array}$$

   (b) $L = \{a^n b^{n+m} c^m \mid n \geq 0 \text{ and } m \geq 0\}$

   Solution: q_0 will be the starting state, z will represent the bottom of the stack, and q_f will be the only final state.

$$\begin{array}{rclrclrcl} \delta(q_0, \lambda, z) &=& \{(q_f, z)\} & \delta(q_1, b, 1) &=& \{(q_1, \lambda)\} & \delta(q_2, c, 0) &=& \{(q_2, \lambda)\} \\ \delta(q_0, a, z) &=& \{(q_0, 1z)\} & \delta(q_1, b, z) &=& \{(q_1, 0z)\} & \delta(q_2, \lambda, z) &=& \{(q_f, z)\} \\ \delta(q_0, a, 1) &=& \{(q_0, 11)\} & \delta(q_1, b, 0) &=& \{(q_1, 00)\} & & & \\ \delta(q_0, b, 1) &=& \{(q_1, \lambda)\} & \delta(q_1, c, 0) &=& \{(q_2, \lambda)\} & & & \\ \delta(q_0, b, z) &=& \{(q_1, 0z)\} & \delta(q_1, \lambda, z) &=& \{(q_f, z)\} & & & \end{array}$$

4. (*10 Points Each*) Consider the following context free grammar, G,

$$\begin{array}{rcl} S &\to& aAbb|bBaa|abAB \\ A &\to& abBA|ab|a \\ B &\to& bBBb|aA|b \end{array}$$

   (a) Convert this grammar to Greibach Normal Form.

   Solution:

$$\begin{array}{rcl} S &\to& aADD|bBCC|aDAB \\ A &\to& aDBA|aD|a \\ B &\to& bBBD|aA|b \\ C &\to& a \\ D &\to& b \end{array}$$

(b) Using the Greibach Normal Form, convert the grammar into a nondeterministic push-down automaton.

Solution: q_0 will be the starting state, z will represent the bottom of the stack, and q_f will be the only final state.

$$\begin{array}{rcl}
\delta(q_0, \lambda, z) & = & \{(q_1, Sz)\} \\
\delta(q_1, a, S) & = & \{(q_1, ADD), (q_1, DAB)\} \\
\delta(q_1, b, S) & = & \{(q_1, BCC)\} \\
\delta(q_1, a, A) & = & \{(q_1, DBA), (q_1, D), (q_1, \lambda)\} \\
\delta(q_1, a, B) & = & \{(q_1, A)\} \\
\delta(q_1, b, B) & = & \{(q_1, BBD), (q_1, \lambda)\} \\
\delta(q_1, a, C) & = & \{(q_1, \lambda)\} \\
\delta(q_1, b, D) & = & \{(q_1, \lambda)\} \\
\delta(q_1, \lambda, z) & = & \{(q_f, z)\}
\end{array}$$

5. (*15 Points*) Construct a deterministic push-down automaton that accepts the language,

$$L = \{wcw^R \mid w \in \{a, b\}^*\}$$

Solution: We will give two possible solutions, the first is similar to the one in the text, except that we can make it deterministic because of the c that separates w from w^R. q_0 will be the starting state, z will represent the bottom of the stack, and q_f will be the only final state.

$$\begin{array}{rclcrcl}
\delta(q_0, a, z) & = & \{(q_0, az)\} & & \delta(q_0, c, z) & = & \{(q_f, z)\} \\
\delta(q_0, b, z) & = & \{(q_0, bz)\} & & \delta(q_0, c, a) & = & \{(q_1, a)\} \\
\delta(q_0, a, a) & = & \{(q_0, aa)\} & & \delta(q_0, c, b) & = & \{(q_1, b)\} \\
\delta(q_0, a, b) & = & \{(q_0, ab)\} & & \delta(q_1, a, a) & = & \{(q_1, \lambda)\} \\
\delta(q_0, b, a) & = & \{(q_0, ba)\} & & \delta(q_1, b, b) & = & \{(q_1, \lambda)\} \\
\delta(q_0, b, b) & = & \{(q_0, bb)\} & & \delta(q_1, \lambda, z) & = & \{(q_f, z)\}
\end{array}$$

The second solution takes the grammar $S \to aSa|bSb|c$, converts it to Greibach Normal Form.

$$\begin{array}{rcl}
S & \to & aSA|bSB|c \\
A & \to & a \\
B & \to & b
\end{array}$$

Then applies the conversion algorithm,

$$\begin{array}{rcl}
\delta(q_0, \lambda, z) & = & \{(q_1, Sz)\} \\
\delta(q_1, a, S) & = & \{(q_1, SA)\} \\
\delta(q_1, b, S) & = & \{(q_1, SB)\} \\
\delta(q_1, c, S) & = & \{(q_1, \lambda)\} \\
\delta(q_1, a, A) & = & \{(q_1, \lambda)\} \\
\delta(q_1, b, B) & = & \{(q_1, \lambda)\} \\
\delta(q_1, \lambda, z) & = & \{(q_f, z)\}
\end{array}$$

6. (*20 Points*) Prove that the language $L = \{w \in \{a, b, c\}^* \mid n_a(w) < n_b(w) < n_c(w)\}$ is not context-free.

Solution: By way of contradiction, assume that L is context-free. Then there exists a positive integer m such that for any word $w \in L$ with $|w| \geq m$, we can write $w = uvxyz$, with $|vxy| \leq m$ and $|vy| \geq 1$, such that $uv^ixy^iz \in L$ for all $i = 0, 1, 2, 3, \ldots$. Let $w = a^m b^{m+1} c^{m+2}$. Note that $w \in L$ and $|w| = 3m+3 \geq m$, so we can write $w = uvxyz$ with the above conditions holding. This will give us five cases for vy, these are a^t, a^tb^r, b^t, b^tc^r, and c^t, where $t > 0$ and $r > 0$.

Case 1: If $vy = a^t$, then uv^2xy^2z is a word with at least $m+1$ a's, hence $n_a(w) \geq n_b(w)$ and thus the word is not in L.

Case 2: If $vy = a^tb^r$, then uv^2xy^2z is a word with at least $m+2$ b's, hence $n_b(w) \geq n_c(w)$ and thus the word is not in L.

Case 3: If $vy = b^t$, then uv^2xy^2z is a word with at least $m+2$ b's, hence $n_b(w) \geq n_c(w)$ and thus the word is not in L.

Case 4: If $vy = b^tc^r$, then uxz is a word with at most m b's, hence $n_a(w) \geq n_b(w)$ and thus the word is not in L.

Case 5: If $vy = c^t$, then uxz is a word with at most $m+1$ c's, hence $n_b(w) \geq n_c(w)$ and thus the word is not in L.

Since all cases have led to a contradiction, the language $L = \{w \in \{a, b, c\}^* \mid n_a(w) < n_b(w) < n_c(w)\}$ is not context-free.