Name: ____

You may fill in the True & False on this page but write all of your responses to the other questions on the extra paper provided. Hand in this exam paper along with your solutions, please place your name on the top of each page.

1. **True & False:** (*3 Points Each*) Mark each of the following as being either true or false.

   (a) ________ The intersection of two context-free languages is context-free.

   (b) ________ The union of two deterministic context-free languages is deterministic context-free.

   (c) ________ The following language is context-free.

   $L = \{w \in \{a,b\}^* \mid n_a(w) < n_b(w)$ and w does not contain any substring of the form $abba^*bb^*\}$

   (d) ________ Any language accepted by a push-down automaton can be represented as a finite union of regular expressions.

   (e) ________ The complement of a deterministic context-free language is context-free, but not necessarily deterministic.

   (f) ________ The language $L = \{P \mid P$ is equivalent to a given program $Q\}$ defines a functional property.

   (g) ________ Gödel numbering is the process of putting the set of all Turing Machines in a one-to-one correspondence with the natural numbers.

   (h) ________ A nondeterministic Turing machine is more powerful than a deterministic Turing machine.

   (i) ________ A countable union of countable sets is countable.

   (j) ________ It is possible for a set to have the same cardinality as a proper subset of itself.

2. **Definitions & Short Answer:** (*5 Points Each*) Answer all of the following.

   (a) What does it mean to parse a word w that is in the language defined by a context-free grammar G?

   (b) What is a derivation tree?

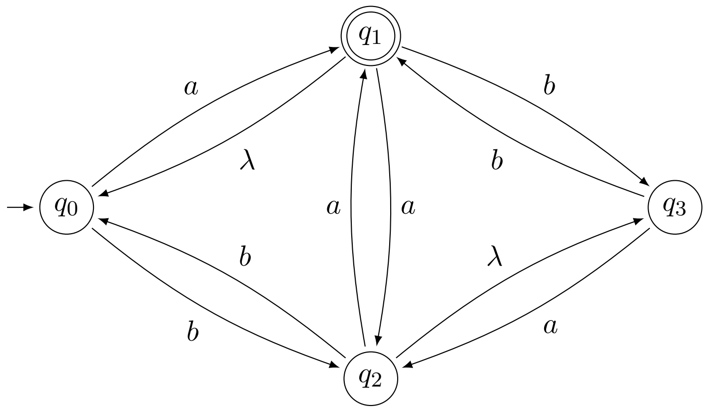
   (c) What does it mean for a language to be ambiguous and inherently ambiguous?

   (d) State Turing's Thesis.

   (e) State Rice's Theorem and briefly discuss its consequences.

   (f) Define an algorithm.

3. **Finite Automata:** (*30 Points*) Consider the following NFA, A.



   (a) Convert this NFA to a DFA.

   (b) Construct a right linear grammar for the language accepted by A.

   (c) Give a regular expression for the language accepted by A.

4. **Context-Free Languages, Grammars & Push-Down Automata:** (*30 Points*) Consider the context-free grammar G with $\Sigma = \{a, b, \}$ and rules,

$$\begin{aligned} S &\rightarrow AB|BA \\ A &\rightarrow aAa|B \\ B &\rightarrow bBbb|a \end{aligned}$$

   (a) Describe $L(G)$.

   (b) Is $L(G)$ regular? Prove your answer.

   (c) Convert the Grammar to Greibach Normal Form.

   (d) Convert the grammar to an NPDA.

5. **Turning Machines:** (*15 Points Each*) Do each of the following.

   (a) Construct a standard Turing Machine by displaying the set of transitions for the Turing Machine that will take as input a unary number, x, and compute $f(x) = x \pmod 6$. For example, an input of $\square\underline{1}111111\square$ will produce an output of $\square\underline{1}\square$, an input of $\square\underline{1}11111\square$ will produce an output of $\square\underline{0}\square$, and an input of $\square\underline{1}11111111111111\square$ will produce an output of $\square\underline{1}11\square$.

   (b) Use the primitives R, L, R_a, L_a, R_b, L_b, $R_\square$, $L_\square$, R_0, L_0, R_1, L_1, $R_{\overline{a}}$, $L_{\overline{a}}$, $R_{\overline{b}}$, $L_{\overline{b}}$, $R_{\overline{\square}}$, $L_{\overline{\square}}$, $R_{\overline{0}}$, $L_{\overline{0}}$, $R_{\overline{1}}$, $L_{\overline{1}}$, a, b, 0, 1, $\square$, A, S, Cp, Shl, Shr, N_L, N_R, W_E, and W_B, and the tape alphabet of $\{a, b, 0, 1, \square\}$ where,

   A — Adds one in binary, the read/write head begins and ends on the leftmost digit. So applying it to $\underline{1}00101$ produces $\underline{1}00110$. Also the number grows to the left, so $\square\underline{1}11$ produces $\underline{1}000$.

   S — Subtracts one in binary, the read/write head begins and ends on the leftmost digit. So applying it to $\underline{1}00110$ produces $\underline{1}00101$. Also the number shrinks on the left, so $\underline{1}000$ produces $\square\underline{1}11$.

Cp — Copies a word. So $\underline{a}ba\square$ produces $\underline{a}ba\square aba$.

Shl — Shifts a word one space to the left. So $\square\underline{a}ba$ produces $\underline{a}ba\square$.

Shr — Shifts a word one space to the right. So $\underline{a}ba\square$ produces $\square\underline{a}ba$.

N_L — Moves the read/write head to the beginning of the next word to the left.

N_R — Moves the read/write head to the beginning of the next word to the right.

W_E — Moves the read/write head to the end of the word. If the read/write head is on a space the head does not move.

W_B — Moves the read/write head to the beginning of the word. If the read/write head is on a space the head does not move.

Construct a Turing machine (in diagram form) that will take an input of a single word from $\{a, b\}^*$ and write the number of a's, in binary, in front of the word and the number of b's, in binary, in back of the word. For example, if the input tape is $\square\underline{a}bbbababbaa\square$ the Turing machine produces $\square 101\square\underline{a}bbbababbaa\square 110\square$. You may use other tape symbols if you would like, in this case the R_x, $R_{\overline{x}}$, L_x and $L_{\overline{x}}$ machines work as usual.

6. **Membership and Non-membership:** (*15 Points Each*) Do any two of the following.

   (a) Prove or disprove that the language $L = \{a^{n^2} \mid n \text{ is an integer}\}$ is regular.

   (b) Prove or disprove that the language $L = \{w \in \{a, b, c\}^* \mid n_a(w) < n_b(w) < n_c(w)\}$ is context-free.

   (c) Prove or disprove that the intersection of two context-free languages is context-free.

   (d) Prove or disprove that the language $L = \{P \mid P \text{ is equivalent to a given program } Q\}$ is undecidable.

7. **General Results:** (*15 Points Each*) Do any two of the following.

   (a) Prove that context-free languages are closed under union, concatenation, and star closure.

   (b) Prove that the cardinality of the set of all Turing machines is countable.

   (c) Prove that the cardinality of the power set of a set A is strictly greater than the cardinality of A.

   (d) Prove that there exists a function $f : \mathbb{N} \to \mathbb{N}$ that is not partial Turing computable.

   (e) Define the Halt (H) program as we did in class and then prove that it does not exist.