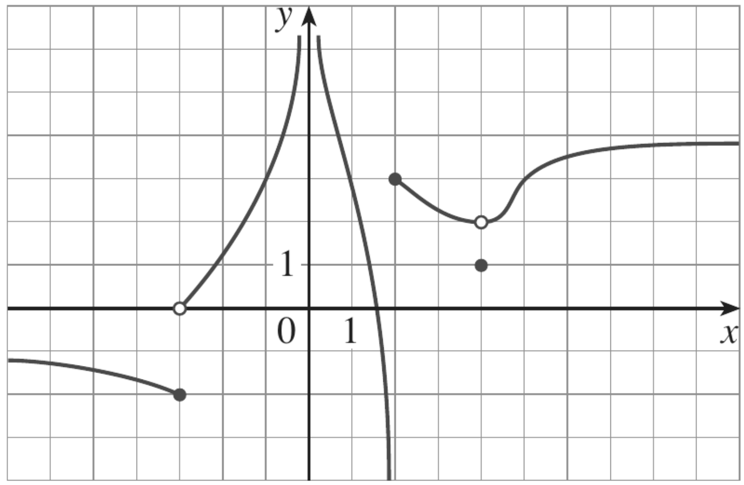
1. (*10 Points*) Given the graph of the the function f below, answer the following.



(a) $\lim_{x\to -3} f(x) =$ Does not exist.

(b) $\lim_{x\to -3^-} f(x) = -2$

(c) $\lim_{x\to -3^+} f(x) = 0$

(d) $\lim_{x\to 0} f(x) = \infty$

(e) $f(0) =$ Does not exist.

(f) $\lim_{x\to 2^+} f(x) = 3$

(g) $\lim_{x\to 2^-} f(x) = -\infty$

(h) $f(2) = 3$

(i) $\lim_{x\to 4} f(x) = 2$

(j) $f(4) = 1$

(k) List all points of discontinuity. For each, state the type of discontinuity and state if the function is continuous from the left or right at that point.

- $x = -3$: Jump, continuous from the left.
- $x = 0$: Infinite, not continuous from the left or right.
- $x = 2$: Infinite, continuous from the right.
- $x = 4$: Removable, not continuous from the left or right.

2. (*10 Points*) Sketch the graph of an example of a function f that satisfies all of the given conditions.

(a) $\lim_{x\to -2^-} f(x) = -1$

(b) $\lim_{x\to -2^+} f(x) = 1$

(c) $f(-2)$ Does not exist.

(d) $\lim_{x\to 0^+} f(x) = -\infty$

(e) $\lim_{x\to 0^-} f(x) = -1$

(f) $f(0) = 0$

(g) $\lim_{x\to 1^-} f(x)$ Does not exist & not infinite.

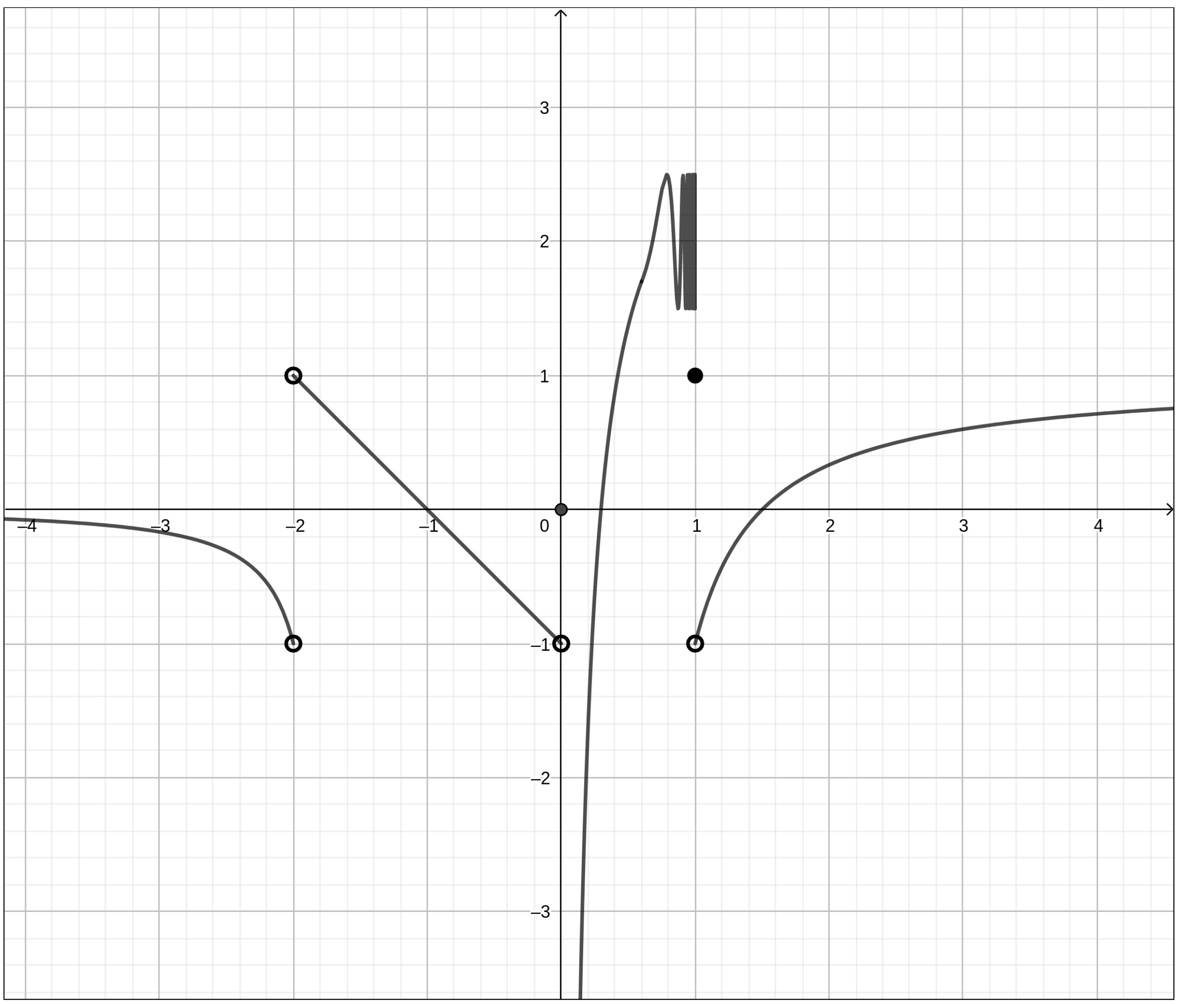
(h) $\lim_{x\to 1^+} f(x) = -1$

(i) $f(1) = 1$

(j) $\lim_{x\to \infty} f(x) = 1$

(k) $\lim_{x\to -\infty} f(x) = 0$

Solution:



3. (*10 Points*) Find the following limit using limit laws. Keep your answer in exact form.

$$\lim_{x\to 3} \frac{x^3-27}{x^2-9}$$

Solution:

$$\begin{aligned}\lim_{x\to 3} \frac{x^3-27}{x^2-9} &= \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)} \\ &= \lim_{x\to 3} \frac{x^2+3x+9}{x+3} = \frac{27}{6} = \frac{9}{2}\end{aligned}$$

4. (*10 Points*) Find the following limit using limit laws. Keep your answer in exact form.

$$\lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

Solution:

$$\begin{aligned}\lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} &= \lim_{h\to 0} \frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h} \\ &= \lim_{h\to 0} \frac{x^2-(x+h)^2}{hx^2(x+h)^2} \\ &= \lim_{h\to 0} \frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2} \\ &= \lim_{h\to 0} \frac{x^2-x^2-2xh-h^2}{hx^2(x+h)^2} \\ &= \lim_{h\to 0} \frac{-2xh-h^2}{hx^2(x+h)^2} \\ &= \lim_{h\to 0} \frac{h(-2x-h)}{hx^2(x+h)^2} \\ &= \lim_{h\to 0} \frac{-2x-h}{x^2(x+h)^2} = \frac{-2x}{x^4} = -\frac{2}{x^3}\end{aligned}$$

5. (*10 Points*) Find the following limit using limit laws. Keep your answer in exact form.

$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$$

Solution:

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} &= \lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} \cdot \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} \\ &= \lim_{x\to 0} \frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})} \\ &= \lim_{x\to 0} \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})} \\ &= \lim_{x\to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}} = 1\end{aligned}$$

6. (*10 Points*) Find the following limit using limit laws. Keep your answer in exact form.

$$\lim_{x\to -5} \frac{2x^2+9x-5}{x^2-25}$$

Solution:

$$\begin{aligned}\lim_{x\to -5} \frac{2x^2+9x-5}{x^2-25} &= \lim_{x\to -5} \frac{(2x-1)(x+5)}{(x+5)(x-5)} \\ &= \lim_{x\to -5} \frac{2x-1}{x-5} = \frac{-11}{-10} = \frac{11}{10}\end{aligned}$$

7. (*10 Points*) Show that there is a solution of the equation

$$\tan(x) = \sin(x) + \frac{1}{2}$$

in the interval $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$

Solution: A solution to

$$\tan(x) = \sin(x) + \frac{1}{2}$$

is equivalent to a solution to

$$\sin(x) + \frac{1}{2} - \tan(x) = 0$$

Let $f(x) = \sin(x) + \frac{1}{2} - \tan(x)$, then $f(\pi/6) = \frac{1}{2} + \frac{1}{2} - \frac{1}{\sqrt{3}} = 1 - \frac{1}{\sqrt{3}} > 0$ and $f(\pi/3) = \frac{\sqrt{3}}{2} + \frac{1}{2} - \sqrt{3} = \frac{\sqrt{3}+1-2\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2} < 0$. Since $f(x)$ is continuous for all real numbers, the intermediate value theorem implies that there is a value $\frac{\pi}{6} < c < \frac{\pi}{3}$ with $f(c) = 0$.

8. (*10 Points*) Find the following limits. Keep your answers in exact form.

$$\lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3} \qquad \lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3}$$

Solution:

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3} &= \lim_{x\to\infty} \frac{\frac{\sqrt{1+4x^6}}{x^3}}{\frac{2-x^3}{x^3}} \\ &= \lim_{x\to\infty} \frac{\sqrt{\frac{1+4x^6}{x^6}}}{\frac{2}{x^3}-1} \\ &= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1} = \frac{2}{-1} = -2\end{aligned}$$

$$\lim_{x\to-\infty}\frac{\sqrt{1+4x^6}}{2-x^3} = \lim_{x\to-\infty}\frac{\frac{\sqrt{1+4x^6}}{x^3}}{\frac{2-x^3}{x^3}}$$
$$= \lim_{x\to-\infty} -\frac{\sqrt{\frac{1+4x^6}{x^6}}}{\frac{2}{x^3}-1}$$
$$= \lim_{x\to-\infty} -\frac{\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1} = -\frac{2}{-1} = 2$$

9. (*10 Points*) Find the following limits. Keep your answers in exact form.

$$\lim_{x\to\infty}\left(\sqrt{25x^2+2}-5x\right) \qquad \lim_{x\to-\infty}\left(\sqrt{25x^2+2}-5x\right)$$

Solution: The second limit is infinite since both terms are increasing without bound and we are adding them together.

$$\lim_{x\to-\infty}\left(\sqrt{25x^2+2}-5x\right) = \infty$$

The first limit is of the indeterminate form $\infty - \infty$.

$$\lim_{x\to\infty}\left(\sqrt{25x^2+2}-5x\right) = \lim_{x\to\infty}\left(\sqrt{25x^2+2}-5x\right)\cdot\frac{\sqrt{25x^2+2}+5x}{\sqrt{25x^2+2}+5x}$$
$$= \lim_{x\to\infty}\frac{25x^2+2-25x^2}{\sqrt{25x^2+2}+5x}$$
$$= \lim_{x\to\infty}\frac{2}{\sqrt{25x^2+2}+5x} = 0$$

10. (*10 Points*) Prove that

$$\lim_{x\to 0} x^4\cos\left(\frac{2}{x}\right) = 0$$

Solution: Since $-1 \leq \cos\left(\frac{2}{x}\right) \leq 1$ we have

$$-x^4 \leq x^4\cos\left(\frac{2}{x}\right) \leq x^4$$

So

$$0 = \lim_{x\to 0} -x^4 \leq \lim_{x\to 0} x^4\cos\left(\frac{2}{x}\right) \leq \lim_{x\to 0} x^4 = 0$$

Hence by the squeeze theorem,

$$\lim_{x\to 0} x^4\cos\left(\frac{2}{x}\right) = 0$$