1. (*15 Points*) Find $f'(x)$ for
$$f(x) = \frac{x}{x+2}$$
using the definition of the derivative.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{x+h}{x+h+2} - \frac{x}{x+2}}{h} \\
&= \lim_{h\to 0} \frac{(x+h)(x+2) - x(x+h+2)}{h(x+h+2)(x+2)} \\
&= \lim_{h\to 0} \frac{x^2+2x+hx+2h-x^2-xh-2x}{h(x+h+2)(x+2)} \\
&= \lim_{h\to 0} \frac{2h}{h(x+h+2)(x+2)} \\
&= \lim_{h\to 0} \frac{2}{(x+h+2)(x+2)} \\
&= \frac{2}{(x+2)^2}
\end{aligned}$$

2. (*15 Points*) Find $f'(x)$ for
$$f(x) = \frac{1}{\sqrt{1+x}}$$
using the definition of the derivative.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{\sqrt{1+x+h}} - \frac{1}{\sqrt{1+x}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{1+x}-\sqrt{1+x+h}}{h\sqrt{1+x+h}\sqrt{1+x}} \\
&= \lim_{h\to 0} \frac{\sqrt{1+x}-\sqrt{1+x+h}}{h\sqrt{1+x+h}\sqrt{1+x}} \cdot \frac{\sqrt{1+x}+\sqrt{1+x+h}}{\sqrt{1+x}+\sqrt{1+x+h}} \\
&= \lim_{h\to 0} \frac{1+x-(1+x+h)}{h\sqrt{1+x+h}\sqrt{1+x}(\sqrt{1+x}+\sqrt{1+x+h})} \\
&= \lim_{h\to 0} \frac{-h}{h\sqrt{1+x+h}\sqrt{1+x}(\sqrt{1+x}+\sqrt{1+x+h})} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{1+x+h}\sqrt{1+x}(\sqrt{1+x}+\sqrt{1+x+h})} \\
&= \frac{-1}{2(1+x)^{3/2}}
\end{aligned}$$

3. (*10 Points*) Using the definition of the derivative, prove that $\frac{d}{dx}(\sin(x)) = \cos(x)$. You may use the facts that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ and $\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)-\sin(x)+\cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)}{h} + \lim_{h\to 0} \frac{\cos(x)\sin(h)}{h} \\
&= \sin(x)\lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x)\lim_{h\to 0} \frac{\sin(h)}{h} \\
&= \sin(x)\cdot 0 + \cos(x)\cdot 1 = \cos(x)
\end{aligned}$$

4. (*10 Points*) Using the definition of the derivative prove the product rule in general, $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$.

Solution: Let $F(x) = f(x)g(x)$

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)+f(x)g(x+h)-f(x)g(x+h)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x+h)-f(x)g(x)+f(x)g(x+h)}{h} \\
&= \lim_{h\to 0} \frac{g(x+h)(f(x+h)-f(x))+f(x)(g(x+h)-g(x))}{h} \\
&= \lim_{h\to 0} g(x+h)\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x)\frac{g(x+h)-g(x)}{h} \\
&= g(x)f'(x) + f(x)g'(x)
\end{aligned}$$

5. (*25 Points*) Using the derivative rules, find the derivatives of each of the following functions. You do not need to simplify your results.

   (a) $f(x) = (x^2+e^x)\sqrt{x}$
   Solution: $f'(x) = (x^2+e^x)\frac{1}{2}x^{-1/2} + (2x+e^x)\sqrt{x}$

(b) $f(x) = \dfrac{6x^4 - 5x}{x+1}$

Solution: $f'(x) = \dfrac{(x+1)(24x^3-5) - (6x^4 - 5x)}{(x+1)^2}$

(c) $f(x) = xe^x \cot(x)$

Solution:

$f'(x) = -(xe^x)\csc^2(x) + (xe^x)' \cot(x) = -(xe^x)\csc^2(x) + (xe^x + e^x)\cot(x)$

(d) $f(x) = \left(\dfrac{x^4+1}{x^2+1}\right)^5$

Solution: $f'(x) = 5\left(\dfrac{x^4+1}{x^2+1}\right)^4 \cdot \dfrac{4x^3(x^2+1) - 2x(x^4+1)}{(x^2+1)^2}$

(e) $f(x) = \sin(x)\cos(1-x^2)$

Solution: $f'(x) = 2x\sin(x)\sin(1-x^2) + \cos(x)\cos(1-x^2)$

6. (*10 Points*) Find a parabola with equation $y = ax^2 + bx + c$ that has slope 4 at $x = 1$, slope -8 at $x = -1$, and passes through the point $(2, 15)$.

Solution: $y' = 2ax + b$ so the information gives us the following equations.

$$\begin{aligned} 4a + 2b + c &= 15 \\ 2a + b &= 4 \\ -2a + b &= -8 \end{aligned}$$

So $4a = 12$ and hence $a = 3$. Then $6 + b = 4$ so $b = -2$. Finally, $12 - 4 + c = 15$ so $c = 7$. Hence the final equation is $y = 3x^2 - 2x + 7$.

7. (*10 Points*) Find all values of x where the following function has a horizontal tangent?

$$f(x) = e^x \cos(x)$$

Solution: $f'(x) = e^x \cos(x) - e^x \sin(x)$, since $e^x > 0$ for all x we need to solve, $\cos(x) - \sin(x) = 0$, which is $\cos(x) = \sin(x)$. The values of x within the interval $[0, 2\pi]$ are $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. So in general our solutions are $x = \frac{\pi}{4} + 2\pi k$ and $x = \frac{5\pi}{4} + 2\pi k$ where k is an integer. We could write these together as $x = \frac{\pi}{4} + \pi k$ where k is an integer.

8. (*10 Points*) At what point on the curve $y = \sqrt{1+2x}$ is the tangent line perpendicular to the line $6x + 2y = 1$?

Solution: $y' = \frac{1}{2}(1+2x)^{-1/2} \cdot 2 = \frac{1}{\sqrt{1+2x}}$. The slope of the given line is -3 so we need to solve, $\frac{1}{\sqrt{1+2x}} = \frac{1}{3}$. Hence $\sqrt{1+2x} = 3$, $1 + 2x = 9$, $2x = 8$, which gives $x = 4$.