1. (*15 Points*) Find $f'(x)$ for
$$f(x) = \frac{1}{1+\sqrt{x}}$$
using the definition of the derivative.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{1+\sqrt{x+h}} - \frac{1}{1+\sqrt{x}}}{h} \\
&= \lim_{h\to 0} \frac{(1+\sqrt{x}) - (1+\sqrt{x+h})}{h(1+\sqrt{x+h})(1+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h(1+\sqrt{x+h})(1+\sqrt{x})} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} \\
&= \lim_{h\to 0} \frac{x-(x+h)}{h(1+\sqrt{x+h})(1+\sqrt{x})(\sqrt{x}+\sqrt{x+h})} \\
&= \lim_{h\to 0} \frac{-h}{h(1+\sqrt{x+h})(1+\sqrt{x})(\sqrt{x}+\sqrt{x+h})} \\
&= \lim_{h\to 0} \frac{-1}{(1+\sqrt{x+h})(1+\sqrt{x})(\sqrt{x}+\sqrt{x+h})} \\
&= \frac{-1}{(1+\sqrt{x})(1+\sqrt{x})(\sqrt{x}+\sqrt{x})} = \frac{-1}{2\sqrt{x}(1+\sqrt{x})^2}
\end{aligned}$$

2. (*15 Points*) Find $f'(x)$ for
$$f(x) = \frac{x+1}{4x-1}$$
using the definition of the derivative.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{x+h+1}{4(x+h)-1} - \frac{x+1}{4x-1}}{h} \\
&= \lim_{h\to 0} \frac{(4x-1)(x+h+1)-(x+1)(4(x+h)-1)}{h(4(x+h)-1)(4x-1)} \\
&= \lim_{h\to 0} \frac{4x^2+4xh+4x-x-h-1-(4x^2+4xh-x+4x+4h-1)}{h(4(x+h)-1)(4x-1)} \\
&= \lim_{h\to 0} \frac{-5h}{h(4(x+h)-1)(4x-1)} \\
&= \lim_{h\to 0} \frac{-5}{(4(x+h)-1)(4x-1)} = -\frac{5}{(4x-1)^2}
\end{aligned}$$

3. (*10 Points*) Using the definition of the derivative, prove that $\frac{d}{dx}(\cos(x)) = -\sin(x)$. You may use the facts that $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$ and $\lim_{x\to 0}\frac{\cos(x)-1}{x} = 0$.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}\\
&= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}\\
&= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\cos(x)-\sin(x)\sin(h)}{h}\\
&= \lim_{h\to 0}\frac{\cos(x)(\cos(h)-1)}{h} - \lim_{h\to 0}\frac{\sin(x)\sin(h)}{h}\\
&= \cos(x)\lim_{h\to 0}\frac{\cos(h)-1}{h} - \sin(x)\lim_{h\to 0}\frac{\sin(h)}{h}\\
&= \cos(x)\cdot 0 - \sin(x)\cdot 1 = -\sin(x)
\end{aligned}$$

4. (*10 Points*) Using the definition of the derivative prove the quotient rule in general, $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$.

Solution: Let $F(x) = \frac{f(x)}{g(x)}$

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}\\
&= \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{hg(x)g(x+h)}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)+f(x)g(x)-f(x)g(x)}{hg(x)g(x+h)}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)-f(x)g(x+h)+f(x)g(x)}{hg(x)g(x+h)}\\
&= \lim_{h\to 0}\frac{g(x)(f(x+h)-f(x))-f(x)(g(x+h)-g(x))}{hg(x)g(x+h)}\\
&= \lim_{h\to 0}\frac{1}{g(x)g(x+h)}\cdot\left(g(x)\frac{f(x+h)-f(x)}{h}-f(x)\frac{g(x+h)-g(x)}{h}\right)\\
&= \frac{1}{g(x)g(x)}\cdot(g(x)f'(x)-f(x)g'(x))\\
&= \frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}
\end{aligned}$$

5. (*25 Points*) Using the derivative rules, find the derivatives of each of the following functions. You do not need to simplify your results.

   (a) $f(x) = e^x(x + x\sqrt{x})$
   Solution: $f'(x) = e^x(x + x\sqrt{x}) + e^x(1 + \frac{3}{2}\sqrt{x})$

   (b) $f(x) = \dfrac{x}{1+\sqrt{x}}$
   Solution: $f'(x) = \dfrac{(1+\sqrt{x}) - x(\frac{1}{2}x^{-1/2})}{(1+\sqrt{x})^2}$

   (c) $f(x) = x\cos(x)\sin(x)$
   Solution: $f'(x) = (x\cos(x))\cos(x) + (x\cos(x))'\sin(x) = (x\cos(x))\cos(x) + (-x\sin(x) + \cos(x))\sin(x) = x\cos^2(x) - x\sin^2(x) + \cos(x)\sin(x)$

   (d) $f(x) = \sin\left(\dfrac{e^x}{1+e^x}\right)$
   Solution: $f'(x) = \cos\left(\dfrac{e^x}{1+e^x}\right) \cdot \dfrac{(1+e^x)e^x - e^x e^x}{(1+e^x)^2} = \cos\left(\dfrac{e^x}{1+e^x}\right) \cdot \dfrac{e^x}{(1+e^x)^2}$

   (e) $f(x) = (1-4x)^2\sqrt{x^2+1}$
   Solution: $f'(x) = (1-4x)^2\frac{1}{2}(x^2+1)^{-1/2}2x + 2(1-4x)(-4)\sqrt{x^2+1} = \frac{x(1-4x)^2}{\sqrt{x^2+1}} - 8(1-4x)\sqrt{x^2+1}$

6. (*10 Points*) Find a parabola with equation $y = ax^2 + bx + c$ that has slope 7 at $x = 1$, slope -17 at $x = -2$, and passes through the point $(1, 8)$.

   Solution: $y' = 2ax + b$ so the information gives us the following equations.

$$\begin{aligned} a+b+c &= 8 \\ 2a+b &= 7 \\ -4a+b &= -17 \end{aligned}$$

   So $6a = 24$ and hence $a = 4$. Then $8 + b = 7$ so $b = -1$. Finally, $4 - 1 + c = 8$ so $c = 5$. Hence the final equation is $y = 4x^2 - x + 5$.

7. (*10 Points*) Find all values of x where the following function has a horizontal tangent?

$$f(x) = x + 2\sin(x)$$

   Solution: $f'(x) = 1 + 2\cos(x)$, so we need to solve, $1 + 2\cos(x) = 0$, which is $\cos(x) = -\frac{1}{2}$. The values of x within the interval $[0, 2\pi]$ are $x = \frac{2\pi}{3}$ and $x = -\frac{2\pi}{3}$. So in general our solutions are $x = \frac{2\pi}{3} + 2\pi k$ and $x = -\frac{2\pi}{3} + 2\pi k$ where k is an integer.

8. (*10 Points*) At what point on the curve $y = \sqrt{4+3x}$ is the tangent line perpendicular to the line $8x + 3y = 7$?

   Solution: $y' = \frac{1}{2}(4+3x)^{-1/2} \cdot 3 = \frac{3}{2\sqrt{4+3x}}$. The slope of the given line is $-\frac{8}{3}$ so we need to solve, $\frac{3}{2\sqrt{4+3x}} = \frac{3}{8}$. Hence $\sqrt{4+3x} = 4$, $4 + 3x = 16$, $3x = 12$, which gives $x = 4$.