1. (*10 Points*) Find $\frac{dy}{dx}$ of
$$\tan(x-y) = 2xy^3 + 1$$
You do not need to simplify your answer.

Solution:

$$\begin{aligned}
\tan(x-y) &= 2xy^3+1 \\
\sec^2(x-y)(1-y') &= 2(3xy^2y' + y^3) \\
\sec^2(x-y) - y'\sec^2(x-y) &= 6xy^2y' + 2y^3 \\
\sec^2(x-y) - 2y^3 &= 6xy^2y' + y'\sec^2(x-y) \\
y' &= \frac{\sec^2(x-y) - 2y^3}{6xy^2 + \sec^2(x-y)}
\end{aligned}$$

2. (*15 Points*) Find the equation of the tangent line, in slope-intercept form, to the conchoid of Nicomedes,
$$x^2y^2 = (y+1)^2(4-y^2)$$
at the point $(2\sqrt{3}, 1)$. Keep your answer in exact form

Solution:

$$\begin{aligned}
x^2y^2 &= (y+1)^2(4-y^2) \\
2x^2yy' + 2xy^2 &= 2(y+1)y'(4-y^2) + (y+1)^2(-2yy')
\end{aligned}$$

You could solve for y' here but the general equation is really not needed, you just want y' at $(2\sqrt{3}, 1)$. Substituting in gives

$$\begin{aligned}
2(2\sqrt{3})^2y' + 2(2\sqrt{3}) &= 2(2)y'(3) + (2)^2(-2y') \\
24y' + 4\sqrt{3} &= 12y' - 8y' \\
24y' - 12y' + 8y' &= -4\sqrt{3} \\
20y' &= -4\sqrt{3} \\
y' &= -\frac{4\sqrt{3}}{20} = -\frac{\sqrt{3}}{5}
\end{aligned}$$

So the equation of the tangent line is

$$\begin{aligned}
y - 1 &= -\frac{\sqrt{3}}{5}(x - 2\sqrt{3}) = -\frac{\sqrt{3}}{5}x + \frac{6}{5} \\
y &= -\frac{\sqrt{3}}{5}x + \frac{11}{5}
\end{aligned}$$

3. (*10 Points Each*) Find $\frac{dy}{dx}$ of the following functions. You do not need to simplify your answer.

(a) $y = (\sin(x))^{\ln(x)}$
Solution:

$$\begin{aligned} y &= (\sin(x))^{\ln(x)} \\ \ln(y) &= \ln((\sin(x))^{\ln(x)}) = \ln(x)\ln(\sin(x)) \\ \frac{y'}{y} &= \frac{1}{x}\ln(\sin(x)) + \ln(x)\frac{1}{\sin(x)}\cos(x) \\ \frac{y'}{y} &= \frac{1}{x}\ln(\sin(x)) + \ln(x)\cot(x) \\ y' &= (\sin(x))^{\ln(x)}\left(\frac{1}{x}\ln(\sin(x)) + \ln(x)\cot(x)\right) \end{aligned}$$

(b) $y = \ln(\tan^{-1}(x^4))$
Solution:

$$\begin{aligned} y &= \ln(\tan^{-1}(x^4)) \\ y' &= \frac{1}{\tan^{-1}(x^4)} \cdot \frac{1}{1+x^8} \cdot 4x^3 \\ &= \frac{4x^3}{\tan^{-1}(x^4)(1+x^8)} \end{aligned}$$

(c) $y = \sqrt{x}e^{x^2-x}(x+1)^{2/3}$
Solution:

$$\begin{aligned} y &= \sqrt{x}e^{x^2-x}(x+1)^{2/3} \\ \ln(y) &= \frac{1}{2}\ln(x) + x^2 - x + \frac{2}{3}\ln(x+1) \\ \frac{y'}{y} &= \frac{1}{2x} + 2x - 1 + \frac{2}{3(x+1)} \\ y' &= \sqrt{x}e^{x^2-x}(x+1)^{2/3}\left(\frac{1}{2x} + 2x - 1 + \frac{2}{3(x+1)}\right) \end{aligned}$$

4. (*15 Points*) The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the logistic function

$$n = f(t) = \frac{a}{1 + be^{-0.7t}}$$

where t is measured in hours. At time $t = 0$ the population is 20 cells and is increasing at a rate of 12 cells/hour. Find the values of a and b. According to this model, what happens to the yeast population in the long run?

Solution: $n = f(t) = \frac{a}{1+be^{-0.7t}} = a(1 + be^{-0.7t})^{-1}$ so

$$f'(t) = -a(1 + be^{-0.7t})^{-2}(-0.7be^{-0.7t}) = \frac{0.7abe^{-0.7t}}{(1 + be^{-0.7t})^2}$$

The exercise gave us that $f(0) = 20$ and $f'(0) = 12$, so we have the following equations,

$$20 = \frac{a}{1 + b} \qquad \text{and} \qquad 12 = \frac{0.7\text{ab}}{(1 + \text{b})^2}$$

From the first equation we have $a = 20(1 + b)$. Substituting that back into the second equation we get,

$$12 = \frac{0.7ab}{(1 + b)^2} = \frac{0.7 \cdot 20(1 + b)b}{(1 + b)^2} = \frac{14b}{1 + b}$$

So

$$\begin{aligned} 12(1 + b) &= 14b \\ 12 + 12b &= 14b \\ 12 &= 2b \\ b &= 6 \end{aligned}$$

Then substituting that back into our equation for a gives $a = 140$. So our equation is

$$n = f(t) = \frac{140}{1 + 6e^{-0.7t}}$$

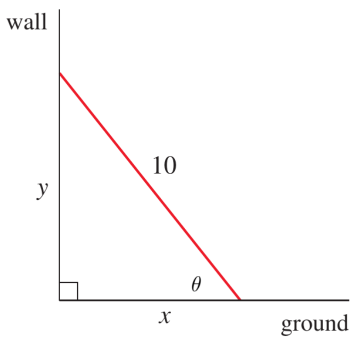
The final part of the exercise asks,

$$\lim_{t\to\infty} \frac{140}{1 + 6e^{-0.7t}} = \frac{140}{1 + 0} = 140$$

So in the long run the population will pan out to 140. This number is called the carrying capacity of the population.

5. (*10 Points*) A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 4 ft/s, how fast is the angle between the ladder and the ground changing when the bottom of the ladder is 6 ft from the wall? Keep your answer in exact form but simplify it down to an "easy" number.

Solution: An image of the situation is below.



The problem gives us that $\frac{dx}{dt} = 4$ and it asks for $\frac{d\theta}{dt}$. An equation that relates x and θ would be $\cos(\theta) = \frac{x}{10}$. Implicitly differentiating both sides with respect to t gives $-\sin(\theta)\frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt}$. So $\frac{d\theta}{dt} = -\frac{4}{10\sin(\theta)}$. When $x = 6$ the Pythagorean theorem gives $y = 8$, so $\sin(\theta) = \frac{8}{10}$. Substituting that into our equation gives,

$$\frac{d\theta}{dt} = -\frac{4}{10\frac{8}{10}} = -\frac{4}{8} = -\frac{1}{2}$$

6. (*10 Points*) Find the differential dy of

$$y = \frac{x+1}{x-1}$$

and evaluate dy when $x = 2$ and $dx = 0.05$. Keep your answer in exact form but simplify it down to an "easy" number.

Solution:

$$dy = \frac{(x-1)-(x+1)}{(x-1)^2}\,dx = -\frac{2}{(x-1)^2}\,dx$$

So for $x = 2$ and $dx = 0.05$ we have $dy = -2 \cdot 0.05 = -0.1$.

7. (*10 Points*) Use a linear approximation (or differentials) to estimate the value if $e^{0.1}$.

Solution: Here we would have $f(x) = e^x$, $a = 0$, and $\Delta x = dx = 0.1$. Using the approximation formula $f(x) \approx f(a) + f'(a)(x-a)$ with $x = 0.1$ gives,

$$e^{0.1} = f(x) \approx f(a) + f'(a)(x-a) = f(0) + f'(0)(0.1-0) = e^0 + e^0 \cdot 0.1 = 1.1$$

Using differentials is similar, here the approximation formula is $f(a+dx) \approx f(a) + dy$. $dy = f'(x)\,dx = e^0 \cdot 0.1 = 0.1$, so $e^{0.1} \approx e^0 + 0.1 = 1.1$.