1. (*50 Points*) For the following function

$$f(x) = \frac{x^2 - x}{2 - x}$$

(a) Find the domain of the function.

Solution: All reals except for $x = 2$.

(b) Find the x and y intercepts of the function.

Solution: y-intercept: $f(0) = 0$. x-intercepts: Solve $f(x) = 0$, clear denominator to obtain $x^2 - x = 0$ and get the solutions $x = 0, 1$.

(c) Test for even/odd symmetry.

Solution: $f(-x) = \frac{x^2+x}{2+x}$ which is neither $f(x)$ nor $-f(x)$. So the graph is not symmetric with respect to the origin or the y-axis.

(d) Find the vertical, horizontal, and tilted asymptotes.

Solution: There is a vertical asymptote at $x = 2$, furthermore,

$$\lim_{x\to 2^+} \frac{x^2 - x}{2 - x} = -\infty \qquad \text{and} \qquad \lim_{x\to 2^-} \frac{x^2 - x}{2 - x} = \infty$$

There are no horizontal asymptotes since

$$\lim_{x\to \infty} \frac{x^2 - x}{2 - x} = -\infty \qquad \text{and} \qquad \lim_{x\to -\infty} \frac{x^2 - x}{2 - x} = \infty$$

There is a tilted asymptote since

$$\frac{x^2 - x}{2 - x} = -x - 1 + \frac{2}{2 - x}$$

the tilted asymptote is $y = -x - 1$.

(e) Find the intervals of increase or decrease.

Solution:

$$f'(x) = \frac{(2 - x)(2x - 1) - (x^2 - x)(-1)}{(2 - x)^2} = \frac{4x - 2 - 2x^2 + x + x^2 - x}{(2 - x)^2} = \frac{-x^2 + 4x - 2}{(2 - x)^2}$$

$f'(x)$ does not exist at $x = 2$ and is 0 when $-x^2 + 4x - 2 = 0$. Solving gives $x = 2-\sqrt{2}$ and $x = 2+\sqrt{2}$. Testing the sign of the derivative gives increasing intervals of $(2-\sqrt{2}, 2) \cup (2, 2+\sqrt{2})$ and decreasing intervals of $(-\infty, 2-\sqrt{2}) \cup (2+\sqrt{2}, \infty)$

(f) Find the local maximum and minimum values.

Solution: From the work above there is a local minimum at $x = 2-\sqrt{2}$ and local maximum at $x = 2+\sqrt{2}$. The actual points are a minimum at $\left(2 - \sqrt{2}, -\frac{3\sqrt{2}-4}{\sqrt{2}}\right)$ and a maximum at $\left(2 + \sqrt{2}, -\frac{3\sqrt{2}+4}{\sqrt{2}}\right)$.

(g) Find the intervals of concave up or concave down.

Solution:

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(\frac{-x^2+4x-2}{(2-x)^2}\right) \\ &= \frac{(2-x)^2(-2x+4)-(-x^2+4x-2)(2(2-x)(-1))}{(2-x)^4} \\ &= \frac{(2-x)(-2x+4)-(-x^2+4x-2)(-2)}{(2-x)^3} \\ &= \frac{-4x+8+2x^2-4x-2x^2+8x-4}{(2-x)^3} \\ &= \frac{4}{(2-x)^3} \end{aligned}$$

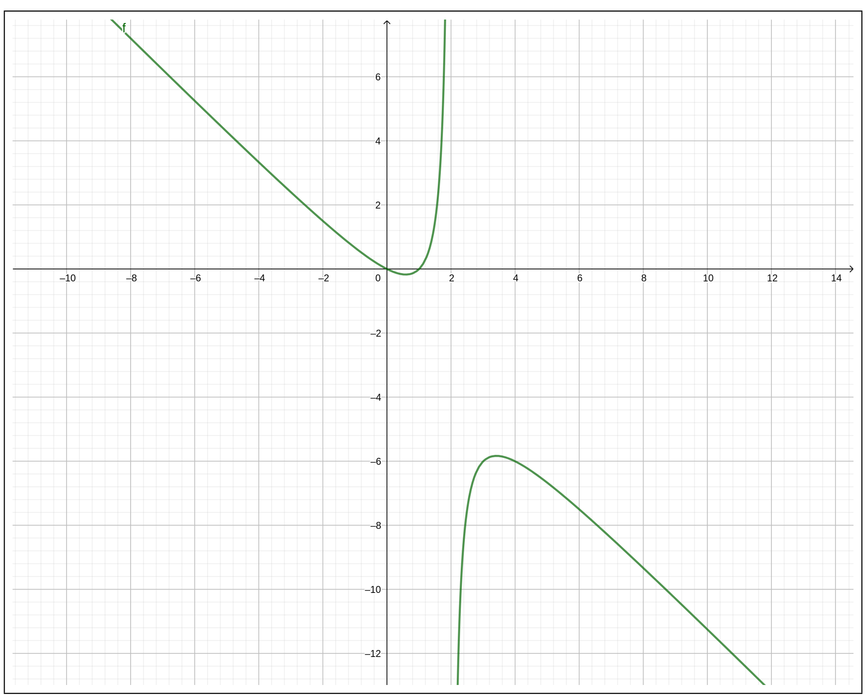
So the only place where the concavity could change is at $x = 2$. Checking the sign of $f''(x)$ we see that the function is concave up on $(-\infty, 2)$ and concave down on $(2, \infty)$.

(h) Find the inflection points.

Solution: Since the only place where the concavity changes is at $x = 2$ and this is where the vertical asymptote is, there is no point of inflection.

(i) Make a sketch of the graph given the information about the properties of the function.

Solution:



2. (*15 Points*) Find the absolute maximum and minimum of

$$f(x) = x^3 - 6x^2 + 5$$

on the interval $[-3, 5]$. Keep your answers in exact form.

Solution: $f'(x) = 3x^2 - 12x$, which gives critical numbers at $x = 0, 4$. Evaluating,

$$\begin{aligned} f(-3) &= -76 \\ f(0) &= 5 \\ f(4) &= -27 \\ f(5) &= -20 \end{aligned}$$

Which gives the absolute minimum at $(-3, -76)$ and the absolute maximum at (0.5).

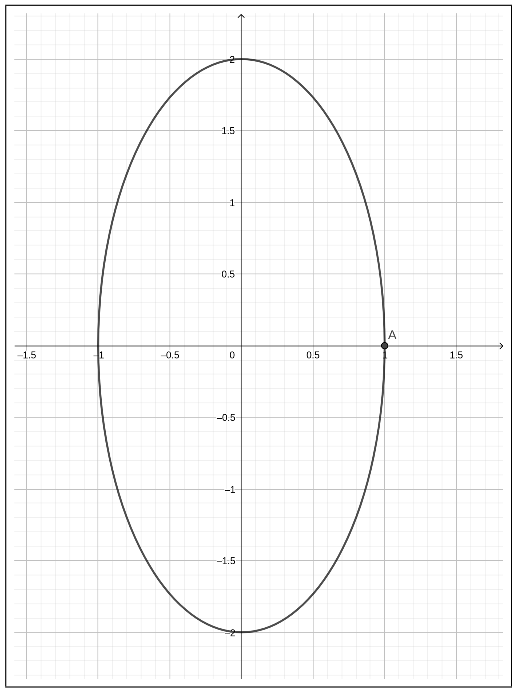
3. (*15 Points*) Find the following limit,

$$\lim_{x\to-\infty} x \ln\left(1 - \frac{1}{x}\right)$$

Solution:

$$\begin{aligned} \lim_{x\to-\infty} x \ln\left(1 - \frac{1}{x}\right) &= \lim_{x\to-\infty} \frac{\ln\left(1 - \frac{1}{x}\right)}{\frac{1}{x}} \\ &= \lim_{x\to-\infty} \frac{\frac{1}{1-\frac{1}{x}} \cdot \frac{1}{x^2}}{-\frac{1}{x^2}} \\ &= \lim_{x\to-\infty} -\frac{1}{1 - \frac{1}{x}} \\ &= -1 \end{aligned}$$

4. (*20 Points*) Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$. Keep your answers in exact form.



Solution: From the equation of the ellipse we have $y = \pm\sqrt{4 - 4x^2}$. If we let (x, y) be any point om the ellipse then the distance between this point and $(1, 0)$ is

$$d = \sqrt{\left(\pm\sqrt{4 - 4x^2} - 0\right)^2 + (x - 1)^2} = \sqrt{4 - 4x^2 + x^2 - 2x + 1} = \sqrt{-3x^2 - 2x + 5}$$

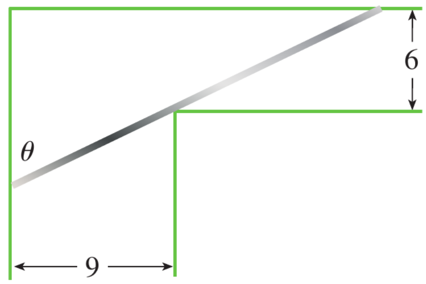
So

$$d' = \frac{-6x - 2}{2\sqrt{-3x^2 - 2x + 5}}$$

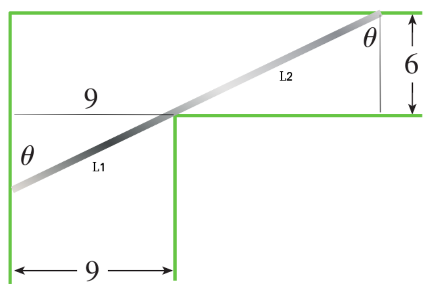
Which gives us one critical number at $x = -\frac{1}{3}$. Checking the sign of d' shows this to be a maximum. So the points that give the maximum distance are

$$\left(-\frac{1}{3}, \frac{4\sqrt{2}}{3}\right) \qquad \text{and} \qquad \left(-\frac{1}{3}, -\frac{4\sqrt{2}}{3}\right)$$

5. **Extra Credit** (*10 Points*): A steel pipe is being carried down a hallway that is 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway, 6 ft wide. What is the length of the longest pipe that can be carried horizontally around the corner? Keep your answers in exact form.



Solution: Let L_1 be the length of the pipe in the 9 foot hallway and let L_2 be the length of the pipe in the 6 foot hallway. Then the total length which we want to maximize is $L = L_1 + L_2$. Construct two right triangles as shown below and note the upper right angle is also θ.



From the augmented image, $L_1 = \frac{9}{\sin(\theta)} = 9\csc(\theta)$ and $L_2 = \frac{6}{\cos(\theta)} = 6\sec(\theta)$. So the function we wish to maximize is

$$L = 9\csc(\theta) + 6\sec(\theta)$$

So

$$L' = -9\csc(\theta)\cot(\theta) + 6\sec(\theta)\tan(\theta) = \frac{-9\cos(\theta)}{\sin^2(\theta)} + \frac{6\sin(\theta)}{\cos^2(\theta)} = \frac{-9\cos^3(\theta) + 6\sin^3(\theta)}{\sin^2(\theta)\cos^2(\theta)}$$

Now L' does not exist if either $\cos(\theta) = 0$ or $\sin(\theta) = 0$. This is at $\theta = 0$ or $\theta = \pi/2$, neither gives a legitimate value for the length L. $L' = 0$ when $-9\cos^3(\theta) + 6\sin^3(\theta) = 0$. Solving this gives us $\tan(\theta) = \sqrt[3]{3/2}$. Creating a triangle with internal angle θ who's tangent is $\sqrt[3]{3/2}$ gives us

$$\sin(\theta) = \frac{\sqrt[3]{3/2}}{\sqrt{1 + \left(\sqrt[3]{3/2}\right)^2}} \qquad \text{and} \qquad \cos(\theta) = \frac{1}{\sqrt{1 + \left(\sqrt[3]{3/2}\right)^2}}$$

So the exact maximum length is

$$L = \frac{9\sqrt{1+\left(\sqrt[3]{3/2}\right)^2}}{\sqrt[3]{3/2}} + 6\sqrt{1+\left(\sqrt[3]{3/2}\right)^2}$$

If one puts all of this back into the first derivative test you actually see that the point is a minimum. This should make sense since if you look at all the lengths of beams the one that just fits around the corner is the one of minimal length of all those that span the two hallways.