1. Optimization (*20 Points*): A rectangular storage container without a lid is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the dimensions of the box that minimizes the cost of materials to create the container.

   Solution: $V = lwh = 10$ and $l = 2w$, so $V = 2w^2h = 10$, and hence $h = \frac{5}{w^2}$. The materials cost is

   $$C(w) = 10lw + 6(2lh + 2wh) = 20w^2 + 6(4wh + 2wh) = 20w^2 + 36wh = 20w^2 + 180/w$$

   So
   $$C'(w) = 40w - 180/w^2 = \frac{40w^3 - 180}{w^2}$$

   The only critical value here is when $40w^3 - 180 = 0$, since $w > 0$. So $w = \sqrt[3]{9/2}$, which is a minimum by the first derivative test, and so $l = 2\sqrt[3]{9/2}$ and $h = 5/\sqrt[3]{(9/2)^2}$.

2. Newton's Method (*20 Points*): Set up the Newton's Method iteration equation for finding all solutions to the equation $2^x = 2 - x^2$. Do not find the solutions.

   Solution: To solve $2^x = 2 - x^2$ we use the function $f(x) = 2^x + x^2 - 2$, $f'(x) = 2^x \ln(2) + 2x$, so the iteration equation is

   $$x_{n+1} = x_n - \frac{2_n^x + x_n^2 - 2}{2_n^x \ln(2) + 2x_n}$$

3. The Definite Integral by Definition (*20 Points*): Using the Definition of a Definite Integral with the right hand endpoint as the test value find the following integral.

   $$\int_0^1 (x^3 - 3x^2)\,dx$$

   Remember that

   $$\sum_{i=1}^n 1 = n \qquad \sum_{i=1}^n i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

   Solution: $\Delta x = 1/n$, $x_n = i/n$, so

   $$\begin{aligned}\int_0^1 (x^3 - 3x^2)\,dx &= \lim_{n\to\infty} \sum_{i=1}^n \left(\left(\frac{i}{n}\right)^3 - 3\left(\frac{i}{n}\right)^2\right) \cdot \frac{1}{n} \\ &= \lim_{n\to\infty} \sum_{i=1}^n \frac{i^3}{n^4} - \frac{3i^2}{n^3} \\ &= \lim_{n\to\infty} \left(\frac{n(n+1)}{2}\right)^2 \cdot \frac{1}{n^4} - \frac{n(n+1)(2n+1)}{6} \cdot \frac{3}{n^3} = \frac{1}{4} - 1 = -\frac{3}{4}\end{aligned}$$

4. The Fundamental Theorem of Calculus (*50 Points*): Find the following,

(a) Given that $y = \displaystyle\int_1^{3x+2} \frac{t}{1+t^3}\, dt$, find $\frac{dy}{dx}$.

Solution: $\dfrac{9x+6}{1+(3x+2)^3}$

(b) Evaluate $\displaystyle\int_1^3 \left(2x + \frac{1}{x}\right)\, dx$.

Solution:

$$\int_1^3 \left(2x + \frac{1}{x}\right)\, dx = x^2 + \ln|x|\Big|_1^3 = 9 + \ln(3) - 1 = 8 + \ln(3)$$

(c) Find the general indefinite integral, $\displaystyle\int \frac{1+\sqrt{x}+x}{x}\, dx$.

Solution:

$$\int \frac{1+\sqrt{x}+x}{x}\, dx = \int \frac{1}{x} + x^{-1/2} + 1\, dx = \ln|x| + 2\sqrt{x} + x + C$$

(d) Evaluate $\displaystyle\int_{\pi/6}^{\pi/3} 4\sec^2(x)\, dx$.

Solution:

$$\int_{\pi/6}^{\pi/3} 4\sec^2(x)\, dx = 4\tan(x)\Big|_{\pi/6}^{\pi/3} = 4(\sqrt{3} - \sqrt{3}/3) = 8\sqrt{3}/3$$

(e) Find the general indefinite integral, $\displaystyle\int \cos^3(x)\sin(x)\, dx$.

Solution: Let $u = \cos(x)$, then $du = -\sin(x)\, dx$ and $dx = -1/\sin(x)\, du$. So

$$\int \cos^3(x)\sin(x)\, dx = \int u^3 \sin(x)\, \frac{-du}{\sin(x)} = -\int u^3\, du = -\frac{u^4}{4} + C = -\frac{\cos^4(x)}{4} + C$$