Exam #1 Key

1. This exercise will be dealing with the function $f(x) = 2x^2 - 4x + 3$ on the interval $[1, 3]$.

   (a) (*10 Points*) Using 4 rectangles Find the right hand Riemann sum that approximates the area under the curve $f(x)$ over the interval $[1, 3]$. Your answer must either be in exact form or correct to all decimal places.
   Solution: $\Delta x = \frac{3-1}{4} = \frac{1}{2}$, $x_i = a + i\Delta x = 1 + \frac{i}{2}$,

$$A \approx f(1.5) \cdot \frac{1}{2} + f(2) \cdot \frac{1}{2} + f(2.5) \cdot \frac{1}{2} + f(3) \cdot \frac{1}{2} = \frac{1}{2}(1.5 + 3 + 5.5 + 9) = \frac{19}{2} = 9.5$$

   (b) (*5 Points*) Using limit and summation notation, write an expression for the exact area under $f(x)$ and over the interval $[1, 3]$. Use the right-hand endpoints as the test values.
   Solution: $\Delta x = \frac{3-1}{n} = \frac{2}{n}$, $x_i = a + i\Delta x = 1 + \frac{2i}{n}$,

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(2\left(1 + \frac{2i}{n}\right)^2 - 4\left(1 + \frac{2i}{n}\right) + 3 \right) \frac{2}{n}$$

   (c) (*5 Points*) Using the limit and sum commands, write a Mathematica command that gives the exact area under $f(x)$ on $[1, 3]$.
   Solution:

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Limit[Sum[(2 (1 + 2 i/n)^2 - 4 (1 + 2 i/n) + 3) 2/n, {i, 1, n}], n -> Infinity]
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   (d) (*10 Points*) Using sum and limit rules, evaluate the limit that gives the exact area under $f(x)$ on $[1, 3]$. Recall the following summation formulas we went over in class.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

   Solution: $\Delta x = \frac{3-1}{n} = \frac{2}{n}$, $x_i = a + i\Delta x = 1 + \frac{2i}{n}$,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2\left(1 + \frac{2i}{n}\right)^2 - 4\left(1 + \frac{2i}{n}\right) + 3 \right) \frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + \frac{8i}{n} + \frac{8i^2}{n^2} - 4 - \frac{8i}{n} + 3 \right) \frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{8i^2}{n^2} \right) \frac{2}{n} \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{2}{n} + \sum_{i=1}^{n} \frac{16i^2}{n^3} \right) \\
&= \lim_{n\to\infty} \left(2 + \frac{16}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right) \right) \\
&= 2 + \frac{16}{3} \\
&= \frac{22}{3}
\end{aligned}$$

2. (*10 Points Each*) Using your integral rules and the Fundamental Theorem of Calculus evaluate the following. Leave your answers in exact form, you do not need to simplify your solutions.

(a) $\displaystyle\int \frac{x^5}{\sqrt[5]{1-3x^3}}\,dx$

Solution: Let $u = 1-3x^3$, then $du = -9x^2\ dx$ and $dx = \frac{du}{-9x^2}$

$$\begin{aligned}\int \frac{x^5}{\sqrt[5]{1-3x^3}}\,dx &= \int \frac{x^5}{\sqrt[5]{u}}\cdot\frac{1}{-9x^2}\,du = -\frac{1}{9}\int\frac{x^3}{\sqrt[5]{u}}\,du = -\frac{1}{9}\int\frac{-\frac{1}{3}(u-1)}{\sqrt[5]{u}}\,du = \frac{1}{27}\int u^{4/5}-u^{-1/5}\,du\\ &= \frac{1}{27}\left(\frac{5}{9}u^{9/5}-\frac{5}{4}u^{4/5}\right)+C = \frac{5}{243}(1-3x^3)^{9/5}-\frac{5}{108}(1-3x^3)^{4/5}+C\end{aligned}$$

(b) $\displaystyle\int \cot(x)\ dx$

Solution: $\cot(x) = \frac{\cos(x)}{\sin(x)}$, so let $u = \sin(x)$, then $du = \cos(x)\ dx$ and $dx = \frac{1}{\cos(x)}\ du$,

$$\int \cot(x)\ dx = \int\frac{\cos(x)}{\sin(x)}\,dx = \int\frac{\cos(x)}{u}\frac{1}{\cos(x)}\,du = \int\frac{1}{u}\,du = \ln|u|+C = \ln|\sin(x)|+C$$

(c) $\displaystyle\int \frac{3}{x}+\frac{x-2}{x^3}-4(x+1)^2\,dx$

Solution:

$$\begin{aligned}\int \frac{3}{x}+\frac{x-2}{x^3}-4(x+1)^2\,dx &= \int\frac{3}{x}+x^{-2}-2x^{-3}-4x^2-8x-4\,dx\\ &= 3\ln|x|-x^{-1}+x^{-2}-\frac{4}{3}x^3-4x^2-4x+C\end{aligned}$$

(d) $\displaystyle\int_3^5 x\sqrt{x^2-5}\,dx$

Solution: Let $u = x^2-5$, $du = 2x\ dx$ and $dx = \frac{du}{2x}$

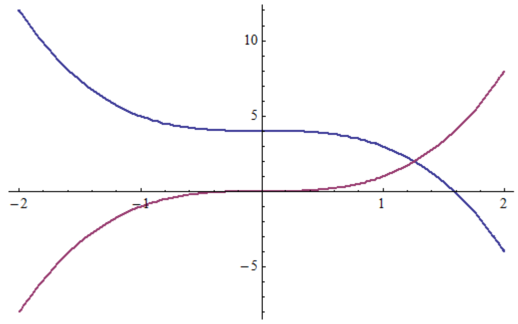
$$\int_3^5 x\sqrt{x^2-5}\,dx = \int_4^{20} x\sqrt{u}\cdot\frac{1}{2x}\,du = \frac{1}{2}\int_4^{20}\sqrt{u}\,du = \frac{1}{3}\left(20^{3/2}-4^{3/2}\right) = \frac{-8+40\sqrt{5}}{3}\approx 27.147573$$

(e) $\displaystyle\frac{d}{dx}\left(\int_{x^3e^x}^{x^2}\sin\left(t^4\right)\ dt\right)$

Solution:

$$\frac{d}{dx}\left(\int_{x^3e^x}^{x^2}\sin\left(t^4\right)\ dt\right) = 2x\sin\left(x^8\right)-\sin\left((x^3e^x)^4\right)\left(x^3e^x+3x^2e^x\right)$$

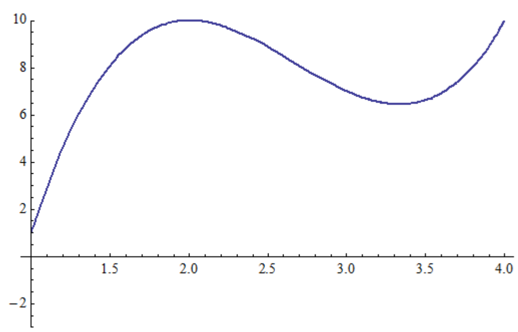
3. (*10 Points*) Find the total area of the region bounded by the curves $y = 4 - x^3$ and $y = x^3$ over the interval $[-2, 2]$. Keep your answer in *exact* form.



Solution: $4 - x^3 = x^3$ when $x = \sqrt[3]{2}$, so

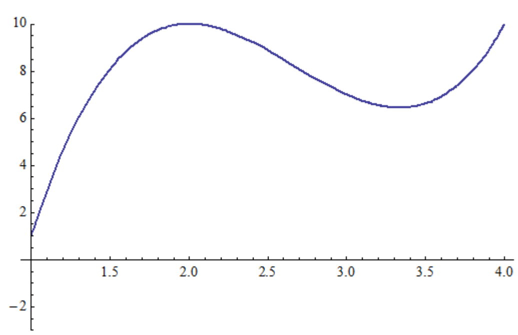
$$\begin{aligned} A &= \int_{-2}^{\sqrt[3]{2}} 4 - x^3 - x^3\,dx + \int_{\sqrt[3]{2}}^{2} x^3 - (4 - x^3)\,dx = \left[4x - \frac{x^4}{2}\right]_{-2}^{\sqrt[3]{2}} + \left[\frac{x^4}{2} - 4x\right]_{\sqrt[3]{2}}^{2} \\ &= \left(\left(4\sqrt[3]{2} - \sqrt[3]{2}\right) - (-8-8)\right) + \left((8-8) - \left(\sqrt[3]{2} - 4\sqrt[3]{2}\right)\right) = 16 + 6\sqrt[3]{2} \approx 23.5595262993692 \end{aligned}$$

4. (*10 Points*) Setup the integral that will find the volume of the solid obtained by revolving region bounded by the curve $f(x) = 3x^3 - 24x^2 + 60x - 38$, $y = 0$, $x = 1$ and $x = 4$ about the line $x = -2$. Do not evaluate the integral.



Solution: $\displaystyle\int_1^4 2\pi(x+2)(3x^3 - 24x^2 + 60x - 38)\,dx = \frac{1053\pi}{5} \approx 661.619412846010456020232 7$

5. (*10 Points*) Setup the integral that will find the volume of the solid obtained by revolving region bounded by the curve $f(x) = 3x^3 - 24x^2 + 60x - 38$, $y = 0$, $x = 1$ and $x = 4$ about the line $y = -1$. Do not evaluate the integral.



Solution: $\displaystyle\int_1^4 \pi(3x^3 - 24x^2 + 60x - 38)^2 - \pi\,dx = \frac{6561\pi}{35} = 588.913982862932383930097 2$