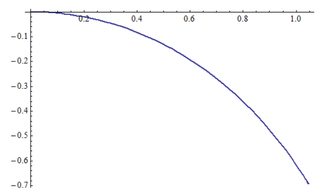
Fall 2013 **Exam #3 Key** Math 202

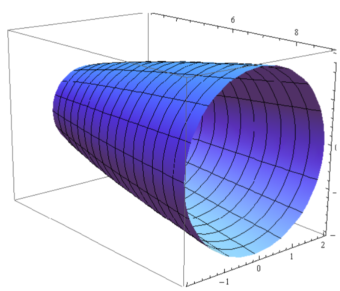
1. (*15 Points*) Find the exact length of the curve $f(x) = \ln(\cos(x))$ for $0 \le x \le \pi/3$. An image of the curve is below.



Solution: With $f(x) = \ln(\cos(x))$ we have $f'(x) = -\tan(x)$

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_0^{\pi/3} \sqrt{1 + \tan^2(x)}\, dx \\ &= \int_0^{\pi/3} \sec(x)\, dx \\ &= \ln(\sec(x) + \tan(x))\Big|_0^{\pi/3} \\ &= \ln(2 + \sqrt{3}) \approx 1.3169578969248167086 \end{aligned}$$

2. (*15 Points*) Find the exact area of the surface obtained by rotating the curve $x = 1 + 2y^2$ about the x-axis for $1 \le y \le 2$. An image of the surface is below.



Solution:

$$\begin{aligned} A &= \int_a^b 2\pi y\, ds \\ &= \int_a^b 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\ &= 2\pi \int_1^2 y\sqrt{1 + 16y^2}\, dy \\ &= \frac{\pi}{24}(1 + 16y^2)^{3/2}\Big|_1^2 \\ &= \frac{\pi}{24}\left(65^{3/2} - 17^{3/2}\right) \approx 59.422434147211626375 \end{aligned}$$

3. (*15 Points*) Do one and only one of the following. Determine if the series converges or diverges, if it converges find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n - 2^n}{5^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n}$

Solution:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n - 2^n}{5^n}$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{3^n - 2^n}{5^n} &= \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n - \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n \\
&= \frac{\frac{3}{5}}{1 - \frac{3}{5}} - \frac{\frac{2}{5}}{1 - \frac{2}{5}} \\
&= \frac{3}{2} - \frac{2}{3} = \frac{5}{6}
\end{aligned}$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n}$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n} &= \sum_{n=1}^{\infty} \frac{1}{n(n+2)} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2} \\
&= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+2}\right) \\
&= \frac{1}{2}\left(\frac{3}{2} + \lim_{n\to\infty}\left(-\frac{1}{n+1} - \frac{1}{n+2}\right)\right) = \frac{3}{4}
\end{aligned}$$

4. (*15 Points*) Do one and only one of the following. Test the series for convergence or divergence. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

Solution:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!}$ Diverges by the ratio test.

$$\lim_{n\to\infty} \frac{\frac{2^{n+1}(n+1)!}{(n+3)!}}{\frac{2^n n!}{(n+2)!}} = \lim_{n\to\infty} \frac{2^{n+1}(n+1)!}{(n+3)!} \cdot \frac{(n+2)!}{2^n n!} = \lim_{n\to\infty} \frac{2(n+1)}{(n+3)} = 2 > 1$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$ Diverges by the root test.

$$\lim_{n\to\infty} \sqrt[n]{\frac{(n!)^n}{n^{4n}}} = \lim_{n\to\infty} \frac{n!}{n^4} = \lim_{n\to\infty} (n-4)! \cdot \frac{n-3}{n} \cdot \frac{n-2}{n} \cdot \frac{n-1}{n} \cdot \frac{n}{n} = \infty > 1$$

Diverges by the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{((n+1)!)^{n+1}}{(n+1)^{4n+4}}}{\frac{(n!)^n}{n^{4n}}} &= \lim_{n\to\infty} \frac{((n+1)!)^{n+1}}{(n+1)^{4n+4}} \cdot \frac{n^{4n}}{(n!)^n} \\
&= \lim_{n\to\infty} \frac{(n+1)^{n+1}(n!)^n n!}{(n+1)^{4n+4}} \cdot \frac{n^{4n}}{(n!)^n} \\
&= \lim_{n\to\infty} \frac{n^{4n}(n+1)^{n+1} n!}{(n+1)^{4n+4}} \\
&= \lim_{n\to\infty} \frac{n^{4n} n!}{(n+1)^{3n+3}} \\
&= \lim_{n\to\infty} \frac{n^{4n}}{(n^{3n+3}+\cdots)} n! = \infty > 1
\end{aligned}$$

5. (*15 Points*) Do one and only one of the following. Test the series for convergence or divergence. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

Solution:

(a) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$ Converges by the integral test.

$$\int_1^{\infty} x^2 e^{-x^3}\, dx = \lim_{t\to\infty} \int_1^t \frac{x^2}{e^{x^3}}\, dx = \lim_{t\to\infty} -\frac{1}{3e^{x^2}}\bigg|_1^t = \lim_{t\to\infty} -\frac{1}{3e^{t^2}} + \frac{1}{3e} = \frac{1}{3e}$$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$ Diverges by the integral test.

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}}\, dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\sqrt{\ln(x)}}\, dx = \lim_{t\to\infty} 2\sqrt{\ln(x)}\bigg|_2^t = \lim_{t\to\infty} 2\sqrt{\ln(t)} - 2\sqrt{\ln(2)} = \infty$$

6. (*15 Points*) Do one and only one of the following. Test the series for convergence or divergence. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \cos(1/n^2)$ (b) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}-1}$

Solution:

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \cos(1/n^2)$ Diverges by the n^{th} term test.

$$\lim_{n\to\infty}\cos(1/n^2) = \cos(0) = 1$$

(b) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}-1}$ Converges by the alternating series test.

Clearly, $b_{n+1} = \dfrac{1}{\sqrt{n+1}-1} < \dfrac{1}{\sqrt{n}-1} = b_n$ and

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}-1} = 0$$

7. (*15 Points*) Do one and only one of the following. Test the series for convergence or divergence. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n^4+1}}{n^3+n^2}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{3n^4+n^2+1}}$

Solution:

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n^4+1}}{n^3+n^2}$ Diverges by limit comparison with $\sum\frac{1}{n}$,

$$\lim_{n\to\infty}\frac{\frac{\sqrt{n^4+1}}{n^3+n^2}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n\sqrt{n^4+1}}{n^3+n^2} = \lim_{n\to\infty}\frac{\sqrt{n^4+1}}{n^2+n} = \lim_{n\to\infty}\frac{\sqrt{n^4+1}}{n^2+n}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n^4}}}{1+\frac{1}{n}} = 1$$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{3n^4+n^2+1}}$ Converges by limit comparison with $\sum\frac{1}{n^{4/3}}$,

$$\lim_{n\to\infty}\frac{\frac{1}{\sqrt[3]{3n^4+n^2+1}}}{\frac{1}{n^{4/3}}} = \lim_{n\to\infty}\frac{n^{4/3}}{\sqrt[3]{3n^4+n^2+1}}\cdot\frac{\frac{1}{n^{4/3}}}{\frac{1}{n^{4/3}}} = \lim_{n\to\infty}\frac{1}{\sqrt[3]{3+\frac{1}{n^2}+\frac{1}{n^4}}} = \frac{1}{\sqrt[3]{3}}$$

8. (*5 Points*) Show that the Harmonic Series is divergent without using the integral test or p-test.
Solution:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \cdots$$

Consider the partial sums,

$$\begin{aligned}
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2} \\
s_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
&> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + \frac{3}{2} \\
s_{16} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} \\
&> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = 1 + \frac{4}{2}
\end{aligned}$$

and in general

$$s_{2^n} > 1 + \frac{n}{2}$$

Since

$$\lim_{n\to\infty} s_{2^n} \geq \lim_{n\to\infty} 1 + \frac{n}{2} = \infty$$

the series diverges.