Name: ____

Write all of your responses on these exam pages, if you need extra space please use the backs of the pages. Show all your work, answers without supporting justification will not receive credit. Keep your answers in exact form and simplify numerical answers to the point where a scientific calculator could get an approximation. Each exercise is worth 20 points.

1. Use the limit of the Riemann sum definition to find the following integral. Evaluate the integral keeping your answer in exact form.

$$\int_{-1}^{2} 4x^2 + x + 2\,dx$$

Solution: $\Delta x = \frac{3}{n}$, $x_i = a + i\Delta x = -1 + \frac{3i}{n}$. So

$$\begin{aligned}
\int_{-1}^{2} 4x^2 + x + 2\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(4\left(-1+\frac{3i}{n}\right)^2 + \left(-1+\frac{3i}{n}\right) + 2\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{15}{n} - \frac{63i}{n^2} + \frac{108i^2}{n^3} \\
&= \lim_{n\to\infty} \frac{15}{n}\cdot n - \frac{63}{n^2}\cdot\frac{n(n+1)}{2} + \frac{108}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} \\
&= \lim_{n\to\infty} 15 - \frac{63}{n^2}\cdot\frac{n^2+n}{2} + \frac{108}{n^3}\cdot\frac{2n^3+\cdots}{6} \\
&= 15 - \frac{63}{2} + 36 = \frac{39}{2}
\end{aligned}$$

2. Using the Fundamental Theorem of Calculus, find

   (a) $\displaystyle \frac{d}{dx}\int_{7}^{x} \sqrt{t+t^3}\,dt$

   Solution: $\sqrt{x+x^3}$

   (b) $\displaystyle \frac{d}{dx}\int_{x^2}^{\sin(x)} e^{-t^2}\,dt$

   Solution: $e^{-\sin^2(x)}\cos(x) - 2xe^{x^4}$

3. Using the Fundamental Theorem of Calculus, find

   (a) $\displaystyle \int_{0}^{4} (4-t)\sqrt{t}\,dt$

   Solution:

$$\int_{0}^{4} (4-t)\sqrt{t}\,dt = \int_{0}^{4} 4t^{1/2} - t^{3/2}\,dt = \frac{8}{3}t^{3/2} - \frac{2}{5}t^{5/2}\Big|_{0}^{4} = \frac{8}{3}4^{3/2} - \frac{2}{5}4^{5/2} = \frac{128}{15}$$

(b) $\displaystyle\int_0^1 e^x + x^e \, dx$

Solution:

$$\int_0^1 e^x + x^e \, dx = e^x + \frac{x^{e+1}}{e+1}\Big|_0^1 = e + \frac{1}{e+1} - 1 = \frac{e^2}{e+1}$$

4. Find the following indefinite integrals

(a) $\displaystyle\int x^2 + 1 + \frac{1}{x^2+1} \, dx$

Solution:

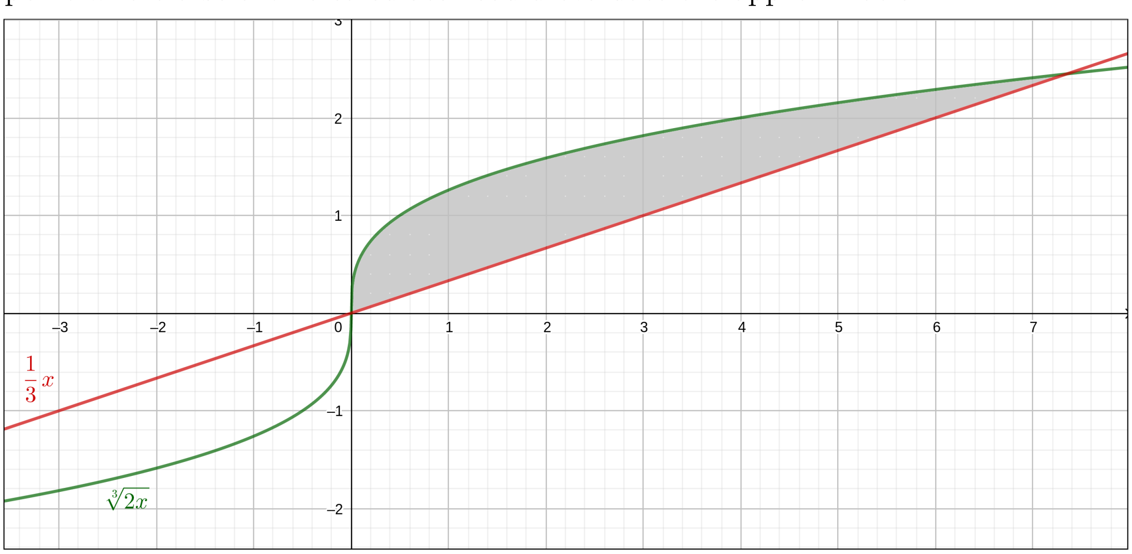
$$\int x^2 + 1 + \frac{1}{x^2+1} = \frac{x^3}{3} + x + \tan^{-1}(x) + C$$

(b) $\displaystyle\int \sin(x)\sqrt{1+\cos(x)} \, dx$

Solution: Let $u = 1 + \cos(x)$, then $du = -\sin(x)\, dx$ and $dx = \frac{du}{-\sin(x)}$

$$\begin{aligned}
\int \sin(x)\sqrt{1+\cos(x)} \, dx &= \int \sin(x)\sqrt{u}\frac{du}{-\sin(x)} \\
&= -\int \sqrt{u} \, du \\
&= -\int u^{1/2} \, du \\
&= -\frac{2}{3}u^{3/2} + C \\
&= -\frac{2}{3}(1+\cos(x))^{3/2} + C
\end{aligned}$$

5. Find the area of the region bounded by $y = \sqrt[3]{2x}$ and $y = \frac{1}{3}x$ in the first quadrant, the shaded region in the image below. Keep your answer in exact form. Simplify to the point where a scientific calculator could evaluate the approximation.



Solution: First find the bounds for the integral. Solving $\frac{1}{3}x = \sqrt[3]{2x}$ gives $\frac{1}{27}x^3 = 2x$, so $0 = \frac{1}{27}x^3 - 2x = x\left(\frac{1}{27}x^2 - 2\right)$. This gives the solutions of $x = 0, \sqrt{54}, -\sqrt{54}$. The two we want are obviously $x = 0$ and $x = \sqrt{54}$.

$$\begin{aligned}
\int_0^{\sqrt{54}} \sqrt[3]{2x} - \frac{1}{3}x\, dx &= \int_0^{\sqrt{54}} \sqrt[3]{2}x^{1/3} - \frac{1}{3}x\, dx \\
&= \frac{3}{4}\sqrt[3]{2}x^{4/3} - \frac{1}{6}x^2\Big|_0^{\sqrt{54}} \\
&= \frac{3}{4}\sqrt[3]{2}x^{4/3} - \frac{1}{6}x^2\Big|_0^{\sqrt{54}} \\
&= \frac{3}{4}\sqrt[3]{2}(\sqrt{54})^{4/3} - \frac{1}{6}(\sqrt{54})^2 - 0 \\
&= \frac{3}{4}\sqrt[3]{2}\cdot 54^{2/3} - 9 \\
&= \frac{3}{4}\sqrt[3]{2\cdot 54^2} - 9 \\
&= \frac{3}{4}\sqrt[3]{5832} - 9 \\
&= \frac{3}{4}\cdot 18 - 9 \\
&= \frac{27}{2} - 9 \\
&= \frac{9}{2}
\end{aligned}$$