1. (*20 Points*) Evaluate the following integral.

$$\int_1^\infty xe^{-x^2}\,dx$$

Solution: Let $u=-x^2$, $du=-2x\,dx$, $dx=-\frac{du}{2x}$

$$\int xe^{-x^2}\,dx=-\frac{1}{2}\int e^u\,du=-\frac{e^u}{2}=-\frac{1}{2e^{x^2}}$$

So

$$\int_1^\infty xe^{-x^2}\,dx=\lim_{t\to\infty}-\frac{1}{2e^{t^2}}+\frac{1}{2e}=\frac{1}{2e}$$

2. (*20 Points*) Find the exact length of the curve $y=\ln(1-x^2)$ for $0\le x\le\frac{1}{2}$.

Solution: $y=\ln(1-x^2)$, so $y'=\frac{-2x}{1-x^2}$

$$\begin{aligned}
L &= \int_0^{1/2}\sqrt{1+\left(\frac{-2x}{1-x^2}\right)^2}\,dx\\
&= \int_0^{1/2}\sqrt{1+\frac{4x^2}{(1-x^2)^2}}\,dx\\
&= \int_0^{1/2}\sqrt{\frac{(1-x^2)^2+4x^2}{(1-x^2)^2}}\,dx\\
&= \int_0^{1/2}\sqrt{\frac{1-2x^2+x^4+4x^2}{(1-x^2)^2}}\,dx\\
&= \int_0^{1/2}\sqrt{\frac{1+2x^2+x^4}{(1-x^2)^2}}\,dx\\
&= \int_0^{1/2}\sqrt{\frac{(1+x^2)^2}{(1-x^2)^2}}\,dx\\
&= \int_0^{1/2}\frac{1+x^2}{1-x^2}\,dx\\
&= \int_0^{1/2}-1+\frac{2}{1-x^2}\,dx\\
&= \int_0^{1/2}-1+\frac{2}{(1+x)(1-x)}\,dx\\
&= \int_0^{1/2}-1+\frac{1}{1+x}+\frac{1}{1-x}\,dx\\
&= -x+\ln(1+x)-\ln(1-x)\Big|_0^{1/2}=-\frac{1}{2}+\ln\left(\frac{3}{2}\right)-\ln\left(\frac{1}{2}\right)=-\frac{1}{2}+\ln(3)
\end{aligned}$$

3. (*15 Points*) Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty} n^2 e^{-n^3}$$

Solution: Series converges by the integral test, let $u = -x^3$, $du = -3x^2\, dx$,

$$\int x^2 e^{-x^3}\, dx = -\frac{1}{3}\int e^u\, du = -\frac{e^{-x^3}}{3} = -\frac{1}{3e^{x^3}}$$

So

$$\int_1^{\infty} x^2 e^{-x^3}\, dx = \lim_{t\to\infty} -\frac{1}{3e^{t^3}} + \frac{1}{3e} = \frac{1}{3e}$$

4. (*15 Points*) Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$$

Solution: Series diverges by direct comparison.

$$\frac{\sqrt{n^4+1}}{n^3+n} > \frac{\sqrt{n^4}}{n^3+n} = \frac{n^2}{n^3+n} > \frac{n^2}{n^3+n^3} = \frac{n^2}{2n^3} = \frac{1}{2n}$$

5. (*15 Points*) Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$$

Solution: Series converges by the ratio test.

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2+1}{5^{n+1}}}{\frac{n^2+1}{5^n}} = \lim_{n\to\infty} \frac{(n+1)^2+1}{5^{n+1}}\frac{5^n}{n^2+1} = \lim_{n\to\infty} \frac{1}{5}\frac{(n+1)^2+1}{n^2+1} = \frac{1}{5}$$

6. (*15 Points*) Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty} \frac{e^n}{n^2}$$

Solution: Series diverges by the divergence test.

$$\lim_{n\to\infty} \frac{e^n}{n^2} = \lim_{n\to\infty} \frac{e^n}{2n} = \lim_{n\to\infty} \frac{e^n}{2} = \infty$$