1. (*20 points*) Find the following indefinite integrals.

(a) $\displaystyle\int \frac{x^5}{\sqrt[3]{1+x^2}}\,dx$ (b) $\displaystyle\int \sqrt{x}\ln(x)\,dx$

Solution: (Part a) Let $u = 1 + x^2$, then $du = 2x\,dx$, and $x^2 = u - 1$, then

$$\begin{aligned}
\int \frac{x^5}{\sqrt[3]{1+x^2}}\,dx &= \frac{1}{2}\int \frac{x^4}{\sqrt[3]{u}}\,du \\
&= \frac{1}{2}\int \frac{(u-1)^2}{\sqrt[3]{u}}\,du \\
&= \frac{1}{2}\int \frac{u^2 - 2u + 1}{\sqrt[3]{u}}\,du \\
&= \frac{1}{2}\int u^{5/3} - 2u^{2/3} + u^{-1/3}\,du \\
&= \frac{1}{2}\left(\frac{3}{8}u^{8/3} - \frac{6}{5}u^{5/3} + \frac{3}{2}u^{2/3}\right) + C \\
&= \frac{3}{16}u^{8/3} - \frac{3}{5}u^{5/3} + \frac{3}{4}u^{2/3} + C \\
&= \frac{3}{16}(1+x^2)^{8/3} - \frac{3}{5}(1+x^2)^{5/3} + \frac{3}{4}(1+x^2)^{2/3} + C
\end{aligned}$$

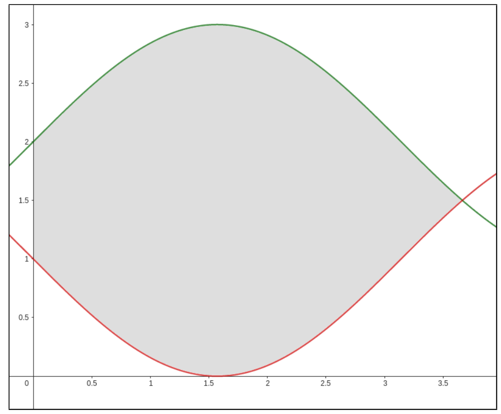
Solution: (Part b) Use parts with $u = \ln(x)$ and $dv = \sqrt{x} = x^{1/2}$. So $du = \frac{1}{x}\,dx$ and $v = \frac{2}{3}x^{3/2}$.

$$\begin{aligned}
\int \sqrt{x}\ln(x)\,dx &= \frac{2}{3}x^{3/2}\ln(x) - \int \frac{2}{3}x^{3/2}\cdot\frac{1}{x}\,dx \\
&= \frac{2x^{3/2}\ln(x)}{3} - \frac{2}{3}\int x^{1/2}\,dx \\
&= \frac{2x^{3/2}\ln(x)}{3} - \frac{4x^{3/2}}{9} + C
\end{aligned}$$

2. (*10 points*) Find the area between the curves $y = 1 - \sin(x)$ and $y = \sin(x) + 2$, in the first quadrant up to the first positive point of intersection. Graph on the right.

3. (*10 points*) Find the volume of the solid produced by taking this region and rotating it about the x-axis. Use the disk and washer method.

4. (*10 points*) Find the volume of the solid produced by taking this region and rotating it about the y-axis. Use the shell method.



Solution: (# 2) First find the intersection point. Setting the equations equal to each other we get $1 - \sin(x) = \sin(x) + 2$, that is, $\sin(x) = -\frac{1}{2}$. The first time this happens with positive x is at $7\pi/6$. So

$$\begin{aligned} A &= \int_0^{7\pi/6} (\sin(x) + 2) - (1 - \sin(x))\, dx \\ &= \int_0^{7\pi/6} 2\sin(x) + 1\, dx \\ &= (-2\cos(x) + x)\Big|_0^{7\pi/6} \\ &= (-2\cos(7\pi/6) + 7\pi/6) - (-2\cos(0) + 0) = 2 + \sqrt{3} + \frac{7\pi}{6} \end{aligned}$$

Solution: (# 3)

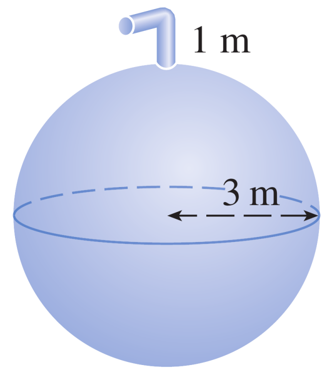
$$\begin{aligned} A &= \int_0^{7\pi/6} \pi(\sin(x) + 2)^2 - \pi(1 - \sin(x))^2\, dx \\ &= \pi \int_0^{7\pi/6} (\sin^2(x) + 4\sin(x) + 4) - (1 - 2\sin(x) + \sin^2(x))\, dx \\ &= \pi \int_0^{7\pi/6} 6\sin(x) + 3\, dx = \pi\,(3x - 6\cos(x))\Big|_0^{7\pi/6} = \pi\left(\frac{7\pi}{2} + 6 + 3\sqrt{3}\right) \end{aligned}$$

Solution: (# 4)

$$\begin{aligned} A &= \int_0^{7\pi/6} 2\pi x((\sin(x) + 2) - (1 - \sin(x)))\, dx = \int_0^{7\pi/6} 2\pi x(2\sin(x) + 1)\, dx \\ &= 4\pi \int_0^{7\pi/6} x\sin(x)\, dx + 2\pi \int_0^{7\pi/6} x\, dx = 4\pi \int_0^{7\pi/6} x\sin(x)\, dx + \frac{49\pi^3}{36} \\ &= 4\pi\,(-x\cos(x) + \sin(x))\Big|_0^{7\pi/6} + \frac{49\pi^3}{36} && \text{Parts} \\ &= 4\pi\left[(-7\pi/6\cos(7\pi/6) + \sin(7\pi/6)) - 0\right] + \frac{49\pi^3}{36} = \frac{7\sqrt{3}\pi^2 - 6\pi}{3} + \frac{49\pi^3}{36} \end{aligned}$$

5. **Extra Credit:** (*5 points*)

The spherical tank to the right is full of water. The tank has a radius of 3 meters and the spout at the top is 1 meter above the top of the tank. The water level is right to the top of the tank but not in the spout. Find the work required to pump the water out of the spout at the top. Remember that the density of water is 1000 kg/m^3 and the force of gravity is 9.8 m/s^2.



Solution: Place an x-axis vertically through the center of the sphere with origin at the center of the sphere. Take a horizontal slice of the tank at a generic x value on the axis. This creates a circle for the intersection. Make a thin (Δx) cylinder from that to approximate the water that all needs to be moved the same distance out the spout. The radius of the cross-sectional circle is $\sqrt{9-x^2}$. So the volume of this slice is,

$$V = \pi r^2 h = \pi(\sqrt{9-x^2})^2 \Delta x = \pi(9-x^2)\Delta x \text{ m}^3$$

So the mass of this slice is

$$m = V \cdot density = 1000\pi(9-x^2)\Delta x \text{ kg}$$

Finally, the force on this slice due to gravity is

$$F = ma = 1000\pi(9-x^2)\Delta x \cdot 9.8 \text{ N} = 9800\pi(9-x^2)\Delta x \text{ N}$$

The distance that this slice must be moved vertically out of the tank is $4-x$ so the total work is

$$\begin{aligned}
\int_{-3}^{3} 9800\pi(9-x^2)(4-x)\,dx &= 9800\pi \int_{-3}^{3} (9-x^2)(4-x)\,dx \\
&= 9800\pi \int_{-3}^{3} 36 - 9x - 4x^2 + x^3\,dx \\
&= 9800\pi \left(36x - \frac{9}{2}x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4\right)\Bigg|_{-3}^{3} \\
&= 9800\pi \Bigg[\left(36\cdot 3 - \frac{9}{2}\cdot 3^2 - \frac{4}{3}\cdot 3^3 + \frac{1}{4}\cdot 3^4\right) \\
&\qquad - \left(36\cdot(-3) - \frac{9}{2}\cdot(-3)^2 - \frac{4}{3}\cdot(-3)^3 + \frac{1}{4}\cdot(-3)^4\right)\Bigg] \\
&= 9800\pi \cdot 144 = 1{,}411{,}200\pi \text{ Nm}
\end{aligned}$$