1. (*10 Points Each*) Find the following integrals.

(a) $\displaystyle\int \frac{2x-3}{x^3+3x}\,dx$

(b) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

(c) $\displaystyle\int \frac{\cos(1/x)}{x^3}\,dx$

(d) $\displaystyle\int x^3(x-1)^{-4}\,dx$

(e) $\displaystyle\int \ln(1+x^2)\,dx$

Solution: (Part a) Partial fractions,

$$\frac{2x-3}{x^3+3x} = \frac{2x-3}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3} = \frac{A}{x} + \frac{Bx+C}{x^2+3} = \frac{-1}{x} + \frac{x+2}{x^2+3}$$

So

$$\begin{aligned}\int \frac{2x-3}{x^3+3x}\,dx &= \int \frac{-1}{x} + \frac{x+2}{x^2+3}\,dx \\ &= \int \frac{-1}{x} + \frac{x}{x^2+3} + \frac{2}{x^2+3}\,dx \\ &= -\ln(x) + \frac{1}{2}\ln(x^2+3) + \frac{2}{\sqrt{3}}\tan^{-1}\left(x/\sqrt{3}\right) + C\end{aligned}$$

Solution: (Part b) Trigonometric substitution, with $x = \sin(\theta)$, so $dx = \cos(\theta)\,d\theta$

$$\begin{aligned}\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{x^2}{\sqrt{1-\sin^2(\theta)}}\cos(\theta)\,d\theta = \int x^2\,d\theta = \int \sin^2(\theta)\,d\theta \\ = \int \frac{1}{2}(1-\cos(2\theta))\,d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C \\ = \frac{1}{2}\theta - \frac{1}{2}\sin(\theta)\cos(\theta) + C = \frac{1}{2}\sin^{-1}(x) - \frac{1}{2}x\sqrt{1-x^2} + C\end{aligned}$$

Solution: (Part c) Let $t = 1/x$, then $dt = -x^{-2}\,dx$ and $dx = -x^2\,dt$. So

$$\begin{aligned}\int \frac{\cos(1/x)}{x^3}\,dx = -\int \frac{\cos(t)}{x}\,dt = -\int \frac{\cos(t)}{1/t}\,dt = -\int t\cos(t)\,dt = -t\sin(t) + \int \sin(t)\,dt \\ = -t\sin(t) - \cos(t) + C = -\frac{1}{x}\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) + C\end{aligned}$$

Solution: (Part d) Let $u = x - 1$, then $du = dx$ and $x = u + 1$. So

$$\int x^3(x-1)^{-4}\,dx = \int (u+1)^3 u^{-4}\,dx = \int \frac{u^3+3u^2+3u+1}{u^4}\,du$$
$$= \int \frac{1}{u} + \frac{3}{u^2} + \frac{3}{u^3} + \frac{1}{u^4}\,du = \ln|u| - \frac{3}{u} - \frac{3}{2u^2} - \frac{1}{3u^3} + C$$
$$= \ln|x-1| - \frac{3}{x-1} - \frac{3}{2(x-1)^2} - \frac{1}{3(x-1)^3} + C$$

Solution: (Part e) Using integration by parts with $u = \ln(1+x^2)$ and $dv = dx$ we get $du = 2x/(1+x^2)\,dx$ and $v = x$. So

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx = x\ln(1+x^2) - \int 2 - \frac{2}{1+x^2}\,dx$$
$$= x\ln(1+x^2) - 2x + 2\tan^{-1}(x) + C$$

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2. **Extra Credit:** (*5 Points*) Find the following integral.

$$\int \frac{1}{\sqrt{\sqrt{x}+1}}\,dx$$

Solution: Let $u = \sqrt{\sqrt{x}+1}$, then $du = \frac{1}{2}(\sqrt{x}+1)^{-1/2} \cdot \frac{1}{2}x^{-1/2}\,dx$, so $dx = 4\sqrt{x}\sqrt{\sqrt{x}+1}\,du$. So

$$\int \frac{1}{\sqrt{\sqrt{x}+1}}\,dx = \int \frac{1}{u} \cdot 4\sqrt{x}\sqrt{\sqrt{x}+1}\,du = 4\int \sqrt{x}\,du = 4\int u^2 - 1\,du$$
$$= \frac{4u^3}{3} - 4u + C = \frac{4\left(\sqrt{\sqrt{x}+1}\right)^3}{3} - 4\sqrt{\sqrt{x}+1} + C$$