1. (*10 Points Each*) Find the following integrals.

(a) $\displaystyle\int \frac{e^{3/x}}{x^3}\,dx$

(b) $\displaystyle\int \frac{x+2}{x^3+3x^2-4x}\,dx$

(c) $\displaystyle\int x^2\tan^{-1}(x)\,dx$

(d) $\displaystyle\int e^{x+e^x}\,dx$

(e) $\displaystyle\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$

Solution: (Part a) Start with the substitution $u = 3/x$ then $du = -3x^{-2}\,dx$ and $dx = -1/3x^2\,du$. After the substitution we will do parts on the result.

$$\int \frac{e^{3/x}}{x^3}\,dx = -\frac{1}{3}\int \frac{e^u}{x^3}\cdot x^2\,du = -\frac{1}{3}\int \frac{e^u}{x}\,du = -\frac{1}{3}\int \frac{e^u}{3/u}\,du = -\frac{1}{9}\int ue^u\,du$$
$$= -\frac{1}{9}\left(ue^u - \int e^u\,du\right) = -\frac{1}{9}\left(ue^u - e^u\right) + C = -\frac{1}{9}\left(\frac{3}{x}e^{3/x} - e^{3/x}\right) + C$$

Solution: (Part b) Start with partial fractions,

$$\frac{x+2}{x^3+3x^2-4x} = \frac{A}{x} + \frac{B}{x+4} + \frac{C}{x-1} = \frac{-1/2}{x} + \frac{-1/10}{x+4} + \frac{3/5}{x-1}$$

So

$$\int \frac{x+2}{x^3+3x^2-4x}\,dx = \int \frac{-1/2}{x} + \frac{-1/10}{x+4} + \frac{3/5}{x-1}\,dx$$
$$= -\frac{1}{2}\ln|x| - \frac{1}{10}\ln|x+4| + \frac{3}{5}\ln|x-1| + C$$

Solution: (Part c) Start with integration by parts, with $u = \tan^{-1}(x)$ and $dv = x^2$, then $du = 1/(x^2+1)$ and $v = \frac{1}{3}x^3$. So

$$\int x^2\tan^{-1}(x)\,dx = \frac{1}{3}x^3\tan^{-1}(x) - \frac{1}{3}\int \frac{x^3}{x^2+1}\,dx = \frac{1}{3}x^3\tan^{-1}(x) - \frac{1}{3}\int x - \frac{x}{x^2+1}\,dx$$
$$= \frac{1}{3}x^3\tan^{-1}(x) - \frac{1}{6}(x^2 - \ln(x^2+1)) + C$$

Solution: (Part d) Do an algebraic step then substitute $u = e^x$.

$$\int e^{x+e^x}\,dx = \int e^x e^{e^x}\,dx = \int e^u\,du = e^u + C = e^{e^x} + C$$

Solution: (Part e) Start with the trigonometric substitution of $x = 2\sin(\theta)$ and hence $dx = 2\cos(\theta)\ d\theta$.

$$\begin{aligned}\int \frac{x^2}{(4-x^2)^{3/2}}\ dx &= \int \frac{4\sin^2(\theta)}{(4-4\sin^2(\theta))^{3/2}} \cdot 2\cos(\theta)\ d\theta = \int \frac{4\sin^2(\theta)}{4^{3/2}\cos^3(\theta)} \cdot 2\cos(\theta)\ d\theta \\ &= \int \frac{\sin^2(\theta)}{\cos^2(\theta)}\ d\theta = \int \tan^2(\theta)\ d\theta = \int \sec^2(\theta) - 1\ d\theta = \tan(\theta) - \theta + C \\ &= \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C\end{aligned}$$

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2. **Extra Credit:** (*5 Points*) Find the following integral.

$$\int \frac{1}{\sqrt{\sqrt{x}+1}}\ dx$$

Solution: Let $u = \sqrt{\sqrt{x}+1}$, then $du = \frac{1}{2}(\sqrt{x}+1)^{-1/2} \cdot \frac{1}{2}x^{-1/2}\ dx$, so $dx = 4\sqrt{x}\sqrt{\sqrt{x}+1}\ du$. So

$$\begin{aligned}\int \frac{1}{\sqrt{\sqrt{x}+1}}\ dx &= \int \frac{1}{u} \cdot 4\sqrt{x}\sqrt{\sqrt{x}+1}\ du = 4\int \sqrt{x}\ du = 4\int u^2 - 1\ du \\ &= \frac{4u^3}{3} - 4u + C = \frac{4\left(\sqrt{\sqrt{x}+1}\right)^3}{3} - 4\sqrt{\sqrt{x}+1} + C\end{aligned}$$