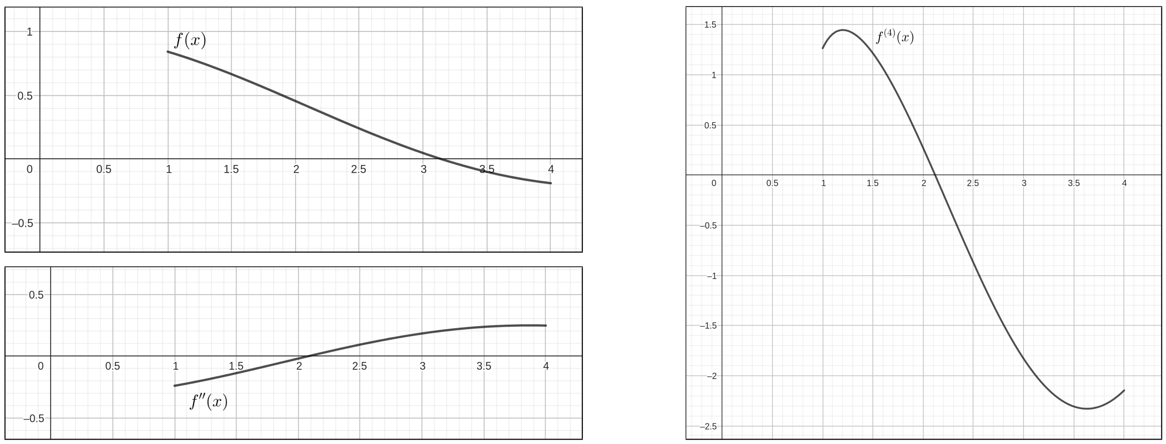
1. (*20 Points*) Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule to approximate

$$\int_1^4 \frac{\sin(x)}{x}\,dx$$

using $n = 6$ divisions, correct to at least 5 decimal places. For each case, state the error bound of the approximation. Graphs of $f(x)$, $f''(x)$, and $f^{(4)}(x)$ are given below.



Solution: $f(x) = \frac{\sin(x)}{x}$

Trapezoidal Rule

$$\begin{aligned}\int_1^4 \frac{\sin(x)}{x}\,dx &\approx \frac{1}{4}\left(f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + f(4)\right)\\ &\approx 0.815992816313824\end{aligned}$$

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \leq \frac{0.25(4-1)^3}{12 \cdot 6^2} = 0.015625$$

Midpoint Rule

$$\begin{aligned}\int_1^4 \frac{\sin(x)}{x}\,dx &\approx \frac{1}{2}\left(f(1.25) + f(1.75) + f(2.25) + f(2.75) + f(3.25) + f(3.75)\right)\\ &\approx 0.8101771710714405\end{aligned}$$

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} \leq \frac{0.25(4-1)^3}{24 \cdot 6^2} = 0.0078125$$

Simpson's Rule

$$\begin{aligned}\int_1^4 \frac{\sin(x)}{x}\,dx &\approx \frac{1}{6}\left(f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)\right)\\ &\approx 0.8120491228884587\end{aligned}$$

$$|E_S| \leq \frac{K(b-a)^5}{180n^4} \leq \frac{2.5(4-1)^5}{180 \cdot 6^4} \approx 0.00260416666666667$$

2. (*15 Points*) Find the following integral if it exists and if it does not show why.

$$\int_{-2}^{5} \frac{1}{(x+2)^{4/3}}\, dx$$

Solution:

$$\int_{-2}^{5} \frac{dx}{(x+2)^{4/3}} = \lim_{t\to -2^+} \int_{t}^{5} \frac{dx}{(x+2)^{4/3}} = \lim_{t\to -2^+} -\frac{3}{\sqrt[3]{x+2}}\Big|_t^5 = \lim_{t\to -2^+} -\frac{3}{\sqrt[3]{7}} + \frac{3}{\sqrt[3]{t+2}} = \infty$$

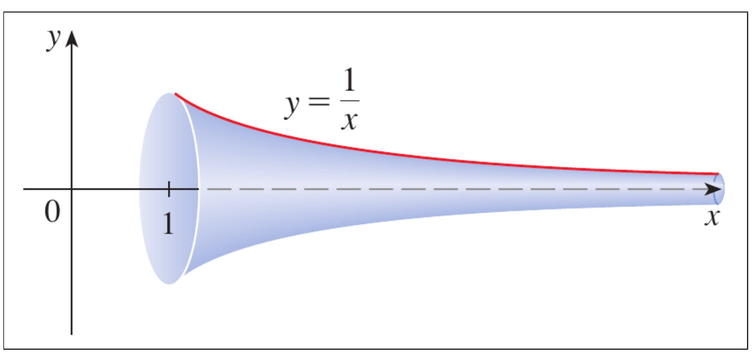
Hence the integral is divergent.

3. (*15 Points*) Find the exact area of the surface obtained by rotating the curve $y = \sqrt{1+e^x}$ about the x-axis on the interval $0 \le x \le 1$.

Solution: $f(x) = \sqrt{1+e^x}$ and $f'(x) = \frac{1}{2}\frac{e^x}{\sqrt{1+e^x}}$

$$\begin{aligned}
A &= \int_0^1 2\pi y\, ds = 2\pi \int_0^1 \sqrt{1+e^x}\sqrt{1+\left(\frac{1}{2}\frac{e^x}{\sqrt{1+e^x}}\right)^2}\, dx \\
&= 2\pi \int_0^1 \sqrt{1+e^x}\sqrt{1+\frac{1}{4}\frac{e^{2x}}{1+e^x}}\, dx = 2\pi \int_0^1 \sqrt{1+e^x+\frac{1}{4}e^{2x}}\, dx \\
&= 2\pi \int_0^1 \sqrt{\left(1+\frac{1}{2}e^x\right)^2}\, dx = 2\pi \int_0^1 1+\frac{1}{2}e^x\, dx = 2\pi\left(x+\frac{1}{2}e^x\right)\Big|_0^1 \\
&= 2\pi\left(1+\frac{1}{2}e-\frac{1}{2}\right) = \pi(e+1)
\end{aligned}$$

4. **Extra Credit** (*5 Points*) The surface formed by rotating the curve $y = 1/x$, $x \ge 1$, about the x-axis is known as Gabriel's horn. Show that the surface area is infinite and that the enclosing volume is finite by finding the exact volume. Hence this is an object that you can fill with paint but you can't paint it.



Solution: The volume, using the disk method is

$$\int_1^{\infty} \pi\frac{1}{x^2}\, dx = \pi \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\, dx = \pi \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t = \pi \lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = \pi$$

The surface area of Gabriel's horn is

$$A = \int_1^\infty 2\pi y \, ds = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \left(-\frac{1}{x^2}\right)^2} \, dx = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}} \, dx$$

$$\geq 2\pi \int_1^\infty \frac{1}{x} \, dx = 2\pi \lim_{t\to\infty} \int_1^t \frac{1}{x} \, dx = 2\pi \lim_{t\to\infty} \ln(x)\Big|_1^t = 2\pi \lim_{t\to\infty} \ln(t) = \infty$$

So you can fill the horn with π gallons of paint but you can't paint it.