1. (*10 Points*) Determine if the following sequence converges or diverges. If it converges, find the value it converges to and if it diverges show why.

$$\left\{e^{2n/(n+2)}\right\}_{n=1}^{\infty}$$

Solution:

$$\lim_{n\to\infty} e^{2n/(n+2)} = e^2$$

2. (*15 Points*) Determine if the following series converges or diverges. If it converges, find the value it converges to and if it diverges show why.

$$\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right)$$

Solution:

$$\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right)=\sum_{n=1}^{\infty}\frac{1}{e^n}+\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum_{n=1}^{\infty}\frac{1}{e^n}+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

The first is geometric with $a=\frac{1}{e}$ and $r=\frac{1}{e}$, so the sum is $\frac{1}{e-1}$. The second is telescoping with sum 1. So the sum of the original series is $1+\frac{1}{e-1}=\frac{e}{e-1}$.

3. (*15 Points*) Use the integral test to determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} ne^{-n}$$

Solution:

$$\begin{aligned}\int_1^{\infty} xe^{-x}\,dx &= \lim_{t\to\infty}\int_1^t xe^{-x}\,dx = \lim_{t\to\infty}\left(-xe^{-x}-e^{-x}\right)\Big|_1^t \qquad \text{(Parts)}\\ &= \lim_{t\to\infty}\left(-te^{-t}-e^{-t}\right)-\left(-2e^{-1}\right)=\frac{2}{e}\end{aligned}$$

Hence the series converges.

4. (*10 Points*) Use comparison or limit comparison to determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{n\sin^2(n)}{1+n^3}$$

Solution:

$$\frac{n\sin^2(n)}{1+n^3}\le\frac{n}{1+n^3}\le\frac{n}{n^3}=\frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, p series with $p>1$, the original series converges.

5. **Extra Credit** (*5 Points*) Show that the following sequence converges and find the limit of the sequence.

$$\left\{\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}, \ldots\right\}$$

Solution: To show that this sequence does converge note that $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2+a_n}$. So $a_2 = \sqrt{2+\sqrt{2}} > \sqrt{2} = a_1$. By induction, assume that $a_n > a_{n-1}$, then $a_{n+1} = \sqrt{2+a_n} > \sqrt{2+a_{n-1}} = a_n$, so the sequence is monotonically increasing. The sequence is also bounded above by 3 (or anything larger). Note that $a_1 = \sqrt{2} < 3$ and by induction if $a_n < 3$ then $a_{n+1} = \sqrt{2+a_n} < \sqrt{2+3} = \sqrt{5} < 3$. So by the Monotonic Sequence Theorem the limit exists.

Since the sequence converges, let x represent the limiting value. Then $x = \sqrt{2+x}$. So $x^2 = 2 + x$, giving $x^2 - x - 2 = 0$. The solutions to this equation are $x = 2$ and $x = -1$. Since all the terms are positive the only viable solution is 2.