1. (*10 Points*) Determine if the following sequence converges or diverges. If it converges, find the value it converges to and if it diverges show why.

$$\left\{ \sqrt[n]{2^{1+3n}} \right\}_{n=1}^{\infty}$$

Solution:

$$\lim_{n\to\infty} \sqrt[n]{2^{1+3n}} = \lim_{n\to\infty} \sqrt[n]{2}\sqrt[n]{8^n} = \lim_{n\to\infty} 8\sqrt[n]{2} = 8$$

2. (*15 Points*) Determine if the following series converges or diverges. If it converges, find the value it converges to and if it diverges show why.

$$\sum_{n=1}^{\infty} \left(\frac{5}{\pi^n} + \frac{1}{n(n+2)} \right)$$

Solution:

$$\sum_{n=1}^{\infty} \left(\frac{5}{\pi^n} + \frac{1}{n(n+2)} \right) = \sum_{n=1}^{\infty} \frac{5}{\pi^n} + \sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \frac{5}{\pi^n} + \sum_{n=1}^{\infty} \left(\frac{1/2}{n} - \frac{1/2}{n+2} \right)$$

The first is geometric with $a = \frac{5}{\pi}$ and $r = \frac{1}{\pi}$, so the sum is $\frac{5}{\pi-1}$. The second is telescoping with sum $\frac{3}{4}$. So the sum of the original series is $\frac{3}{4} + \frac{5}{\pi-1}$.

3. (*15 Points*) Use the integral test to determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$$

Solution:

$$\begin{aligned}\int_2^{\infty} \frac{\ln(x)}{x^2}\,dx &= \lim_{t\to\infty} \int_2^t \frac{\ln(x)}{x^2}\,dx = \lim_{t\to\infty} \left(-\frac{\ln(x)}{x} - \frac{1}{x} \right)\Bigg|_2^t \qquad \text{(Parts)} \\ &= \lim_{t\to\infty} \left(-\frac{\ln(t)}{t} - \frac{1}{t} \right) + \frac{1}{2}(\ln(2)+1) = \frac{1}{2}(\ln(2)+1)\end{aligned}$$

Hence the series converges.

4. (*10 Points*) Use comparison or limit comparison to determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2-1}}$$

Solution: Limit compare with $1/n^2$.

$$\lim_{n\to\infty} \frac{1/n^2}{1/(n\sqrt{n^2-1})} = \lim_{n\to\infty} \frac{n\sqrt{n^2-1}}{n^2} = \lim_{n\to\infty} \frac{n^2\sqrt{1-1/n^2}}{n^2} = \lim_{n\to\infty} \sqrt{1-1/n^2} = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, p series with $p > 1$, the original series converges.

5. **Extra Credit** (*5 Points*) Show that the following sequence converges and find the limit of the sequence.

$$\left\{ \sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}, \ldots \right\}$$

Solution: To show that this sequence does converge note that $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2+a_n}$. So $a_2 = \sqrt{2+\sqrt{2}} > \sqrt{2} = a_1$. By induction, assume that $a_n > a_{n-1}$, then $a_{n+1} = \sqrt{2+a_n} > \sqrt{2+a_{n-1}} = a_n$, so the sequence is monotonically increasing. The sequence is also bounded above by 3 (or anything larger). Note that $a_1 = \sqrt{2} < 3$ and by induction if $a_n < 3$ then $a_{n+1} = \sqrt{2+a_n} < \sqrt{2+3} = \sqrt{5} < 3$. So by the Monotonic Sequence Theorem the limit exists.

Since the sequence converges, let x represent the limiting value. Then $x = \sqrt{2+x}$. So $x^2 = 2+x$, giving $x^2 - x - 2 = 0$. The solutions to this equation are $x = 2$ and $x = -1$. Since all the terms are positive the only viable solution is 2.