1. (*10 Points Each*) For each of the following series determine if the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$ (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos(1/n^2)$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n+4^n}$

Solution:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$
Using the root test

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n^{2n}}{(1+n)^{3n}}\right|} = \lim_{n\to\infty} \frac{n^2}{(1+n)^3} = 0 < 1$$

So the series is absolutely convergent.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos(1/n^2)$
Using the divergence test $\lim_{n\to\infty} \cos(1/n^2) = \cos(0) = 1 \neq 0$, so the series is divergent.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n+4^n}$
Using direct comparison,

$$\frac{5^n}{3^n+4^n} > \frac{5^n}{4^n+4^n} = \frac{5^n}{2\cdot 4^n} = \frac{1}{2}\cdot\left(\frac{5}{4}\right)^n$$

the series $\sum_{n=1}^{\infty} \left(\frac{5}{4}\right)^n$ is a geometric series with $r > 1$ and hence diverges. By the comparison above the original series will also diverge.

2. (*10 Points*) Find the interval and radius of convergence of the following power series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}(x-1)^n$$

Solution: Using the ratio test,

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{(2(n+1)-1)2^{n+1}}(x-1)^{n+1}}{\frac{(-1)^n}{(2n-1)2^n}(x-1)^n} \right| = \lim_{n\to\infty} \left(\frac{(2n-1)2^n}{(2n+1)2^{n+1}}\right)|x-1| = \frac{1}{2}|x-1|$$

So we have convergence when $\frac{1}{2}|x-1| < 1$, specifically $-1 < x < 3$. Checking the endpoints, first $x = -1$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}(-2)^n = \sum_{n=1}^{\infty} \frac{1}{2n-1}$$

which diverges by comparison to the harmonic series, second $x = 3$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n} 2^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$$

which converges by the alternating series test. So the interval of convergence is $(-1, 3]$ and the radius of convergence is $R = 2$.

3. (*10 Points*) Given that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

Find the power series for the following function as well as its radius of convergence.

$$f(x) = \frac{x}{(3-2x)^2}$$

Solution: Recall that

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=1}^{\infty} n x^{n-1}$$

So

$$f(x) = \frac{x}{(3-2x)^2} = \frac{x}{9} \cdot \frac{1}{\left(1 - \frac{2}{3}x\right)^2} = \frac{x}{9} \sum_{n=1}^{\infty} n \left(\frac{2}{3}x\right)^{n-1} = \sum_{n=1}^{\infty} \frac{n\, 2^{n-1}}{3^{n+1}} x^n$$

The radius of convergence is $R = \frac{3}{2}$.

4. **Extra Credit** (*5 Points*) Find the interval and radius of convergence of the following power series.

$$\sum_{n=1}^{\infty} \frac{n! x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

Solution: Using the ratio test,

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)! x^{n+1}}{1\cdot 3\cdot 5\cdots(2n+1)}}{\frac{n! x^n}{1\cdot 3\cdot 5\cdots(2n-1)}} \right| = \lim_{n\to\infty} \frac{n+1}{2n+1} |x| = \frac{1}{2}|x|$$

So we have convergence for $-2 < x < 2$. Checking the endpoints, first $x = -2$,

$$\sum_{n=1}^{\infty} \frac{n!(-2)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \sum_{n=1}^{\infty} \frac{(-1)^n n! 2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

Note that

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{(-1)^n n! 2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \right| &= \lim_{n\to\infty} \frac{n! 2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \lim_{n\to\infty} \frac{(1 \cdot 2 \cdot 3 \cdots n) 2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \\
&= \lim_{n\to\infty} \left(1 \cdot \frac{2}{3} \cdot \frac{3}{5} \cdot \frac{4}{7} \cdots \frac{n}{2n-1}\right) 2^n > \lim_{n\to\infty} \left(\frac{1}{2}\right)^n 2^n = 1
\end{aligned}$$

So by the divergence test the series diverges. The same will hold true for $x = 2$. So the interval of convergence is $(-2, 2)$, and the radius of convergence is $R = 2$.