1. (*10 Points Each*) For each of the following series determine if the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^4}{4^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} n\sin(1/n)$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n^4+1}}{n^3+n}$

Solution:

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^4}{4^n}$

Using the ratio test

$$\lim_{n\to\infty}\left|\frac{(-1)^n\frac{(n+1)^4}{4^{n+1}}}{(-1)^{n-1}\frac{n^4}{4^n}}\right| = \lim_{n\to\infty}\frac{(n+1)^4}{4n^4} = \frac{1}{4} < 1$$

So the series is absolutely convergent.

(b) $\displaystyle\sum_{n=1}^{\infty} n\sin(1/n)$

Using the divergence test

$$\lim_{n\to\infty} n\sin(1/n) = \lim_{n\to\infty}\frac{\sin(1/n)}{1/n} = \lim_{n\to\infty}\frac{\cos(1/n)(-1/n^2)}{-1/n^2} = 1 \neq 0$$

So the series is divergent.

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n^4+1}}{n^3+n}$

Using direct comparison,

$$\frac{\sqrt{n^4+1}}{n^3+n} > \frac{n^2}{n^3+n} > \frac{n^2}{n^3+n^3} = \frac{1}{2n}$$

the series $\sum_{n=1}^{\infty}\frac{1}{2n}$ is just half the harmonic series and hence diverges. By the comparison above the original series will also diverge.

2. (*10 Points*) Find the interval and radius of convergence of the following power series.

$$\sum_{n=1}^{\infty}\frac{\sqrt{n}}{8^n}(x+6)^n$$

Solution: Using the ratio test,

$$\lim_{n\to\infty}\left|\frac{\frac{\sqrt{n+1}}{8^{n+1}}(x+6)^{n+1}}{\frac{\sqrt{n}}{8^n}(x+6)^n}\right| = \lim_{n\to\infty}\left(\frac{\sqrt{n+1}\cdot 8^n}{\sqrt{n}\cdot 8^{n+1}}\right)|x+6| = \frac{1}{8}|x+6|$$

So we have convergence when $\frac{1}{8}|x+6|<1$, specifically $-14<x<2$. Checking the endpoints, first $x=-14$,

$$\sum_{n=1}^{\infty}\frac{\sqrt{n}}{8^n}(-8)^n=\sum_{n=1}^{\infty}(-1)^n\sqrt{n}$$

which diverges by the divergence test, second $x=2$,

$$\sum_{n=1}^{\infty}\frac{\sqrt{n}}{8^n}(8)^n=\sum_{n=1}^{\infty}\sqrt{n}$$

which diverges also by the divergence test. So the interval of convergence is $(-14,2)$ and the radius of convergence is $R=8$.

3. (*10 Points*) Given that

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots=\sum_{n=0}^{\infty}x^n$$

Find the power series for the following function as well as its radius of convergence.

$$f(x)=\frac{x^2}{(1+2x)^2}$$

Solution: Recall that

$$\frac{1}{(1-x)^2}=\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{d}{dx}\left(\sum_{n=0}^{\infty}x^n\right)=\sum_{n=1}^{\infty}nx^{n-1}$$

So

$$f(x)=\frac{x^2}{(1+2x)^2}=x^2\cdot\frac{1}{\left(1-(-2x)\right)^2}=x^2\sum_{n=1}^{\infty}n\left(-2x\right)^{n-1}=\sum_{n=1}^{\infty}(-1)^{n-1}n2^{n-1}x^{n+1}$$

The radius of convergence is $R=\frac{1}{2}$.

4. **Extra Credit** (*5 Points*) Find the interval and radius of convergence of the following power series.

$$\sum_{n=1}^{\infty}\frac{n!x^n}{1\cdot3\cdot5\cdots(2n-1)}$$

Solution: Using the ratio test,

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)!x^{n+1}}{1\cdot3\cdot5\cdots(2n+1)}}{\frac{n!x^n}{1\cdot3\cdot5\cdots(2n-1)}}\right|=\lim_{n\to\infty}\frac{n+1}{2n+1}\left|x\right|=\frac{1}{2}|x|$$

So we have convergence for $-2 < x < 2$. Checking the endpoints, first $x = -2$,

$$\sum_{n=1}^{\infty} \frac{n!(-2)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \sum_{n=1}^{\infty} \frac{(-1)^n n! 2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

Note that

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{(-1)^n n! 2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \right| &= \lim_{n\to\infty} \frac{n! 2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \lim_{n\to\infty} \frac{(1 \cdot 2 \cdot 3 \cdots n) 2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \\ &= \lim_{n\to\infty} \left(1 \cdot \frac{2}{3} \cdot \frac{3}{5} \cdot \frac{4}{7} \cdots \frac{n}{2n-1} \right) 2^n > \lim_{n\to\infty} \left(\frac{1}{2} \right)^n 2^n = 1\end{aligned}$$

So by the divergence test the series diverges. The same will hold true for $x = 2$. So the interval of convergence is $(-2, 2)$, and the radius of convergence is $R = 2$.