1. (*25 Points*): Determine whether the integral

$$\int_1^\infty \frac{1}{x^2+x}\,dx$$

is convergent or divergent. If it is convergent, find its value.

Solution:

$$\int_1^\infty \frac{1}{x^2+x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+x}\,dx = \lim_{t\to\infty}\int_1^t -\frac{1}{x+1}+\frac{1}{x}\,dx$$
$$= \lim_{t\to\infty}\left(\ln(x)-\ln(x+1)\right)\Big|_1^t = \lim_{t\to\infty}\left(\ln(t)-\ln(t+1)\right)+\ln(2)$$
$$= \lim_{t\to\infty}\ln\left(\frac{t}{t+1}\right)+\ln(2) = \ln(2)$$

2. (*25 Points*): Find the exact length of the curve,

$$f(x) = \frac{x^2}{4}-\frac{\ln(x)}{2}$$

on $1 \le x \le 2$.

Solution: $f'(x) = \frac{x}{2}-\frac{1}{2x}$, so

$$L = \int_1^2 \sqrt{1+\left(\frac{x}{2}-\frac{1}{2x}\right)^2}\,dx = \int_1^2 \sqrt{\left(\frac{x}{2}+\frac{1}{2x}\right)^2}\,dx$$
$$= \int_1^2 \frac{x}{2}+\frac{1}{2x}\,dx = \frac{x^2}{4}+\frac{1}{2}\ln(x)\Big|_1^2 = 1+\frac{1}{2}\ln(2)-\frac{1}{4} = \frac{3}{4}+\frac{1}{2}\ln(2)$$

3. (*25 Points*): Find the exact area of the surface obtained by rotating the curve $y^2 = x+1$, $0 \le x \le 3$, about the x-axis.

Solution: $x = y^2-1$, so $dx/dy = 2y$, and

$$A = \int_a^b 2\pi y\,ds = \int_1^2 2\pi y\sqrt{1+(2y)^2}\,dy = 2\pi\int_1^2 y\sqrt{4y^2+1}\,dy$$
$$= 2\pi\cdot\frac{(4y^2+1)^{3/2}}{12}\Bigg|_1^2 = \frac{\pi}{6}\left(17^{3/2}-5^{3/2}\right)$$

4. (*25 Points*): Determine whether the sequence

$$a_n = n^2e^{-n}$$

converges or diverges. If it converges, find the limit.

Solution:

$$\lim_{n\to\infty} n^2e^{-n} = \lim_{n\to\infty}\frac{n^2}{e^n} = \lim_{n\to\infty}\frac{2n}{e^n} = \lim_{n\to\infty}\frac{2}{e^n} = 0$$

5. **Extra Credit**: (*10 Points*): **Do one and only one of the following.**

(a) A sequence $\{a_n\}$ is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. That is, the sequence is $\sqrt{2}$, $\sqrt{2+\sqrt{2}}$, $\sqrt{2+\sqrt{2+\sqrt{2}}}$, $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$, $\ldots$.

i. Show that $\{a_n\}$ is increasing and bounded above by 3, hence it converges.

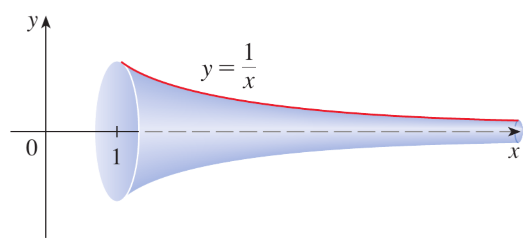
ii. Find $\lim_{n\to\infty} a_n$.

Solution: Note that in the innermost root we always have $2+\sqrt{2}$, and $2+\sqrt{2} > 2$. If we replace the innermost root $2+\sqrt{2}$ with 2 we thus get a smaller value, but this is just the previous term in the sequence. Hence $a_{n+1} > a_n$, therefore the sequence is increasing.

To verify that the sequence is bounded by 3, we use induction. First, $a_1 = \sqrt{2} < 3$, now assume that $a_n < 3$, then $a_{n+1} = \sqrt{2+a_n} < \sqrt{2+3} = \sqrt{5} < 3$, hence all values in the sequence are less than 3.

For the limit, note that if we denote the value of the limit as x then $x = \sqrt{2+x}$. So $x^2 = 2+x$ and the sum is a solution to $x^2-x-2=0$, that is $(x-2)(x+1)=0$. So the sum is either 2 or -1, since -1 is clearly an extraneous solution the sum is 2.

(b) **Gabriel's Horn**: The surface formed by rotating the curve $y = 1/x$, $x \geq 1$, about the x-axis is known as Gabriel's horn. Show that the surface area is infinite and show that the volume is finite. Hence this is an object you can fill with paint but you cannot paint it.



Solution:

$$A = \int_1^\infty 2\pi \frac{1}{x}\sqrt{1+\left(\frac{-1}{x^2}\right)^2}\,dx = 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx \geq 2\pi\int_1^\infty \frac{1}{x}\,dx = \infty$$

$$V = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi\int_1^\infty \frac{1}{x^2}\,dx = \pi\lim_{t\to\infty} -\frac{1}{t}+1 = \pi$$