1. (*45 Points*): Test the series for convergence or divergence.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

   Solution: Converges, using the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{\frac{3^{n+1}(n+1)^2}{(n+1)!}}{\frac{3^n n^2}{n!}} \right| = \lim_{n\to\infty} \left| \frac{3(n+1)^2}{n^2(n+1)} \right| = 0$$

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

   Solution: Diverges, using the Integral Test.

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}}\, dx = \lim_{t\to\infty} 2\sqrt{\ln(x)}\Big|_2^t = \lim_{t\to\infty} 2\sqrt{\ln(t)} - 2\sqrt{\ln(2)} = \infty$$

   (c) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$

   Solution: Diverges, using the Divergence Test.

$$\lim_{n\to\infty} \ln\left(\frac{n}{3n+1}\right) = \ln(1/3) \neq 0$$

2. (*20 Points*): Find the radius of convergence and interval of convergence of the power series,

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}} x^n$$

Solution:

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}4^{n+1}}{\sqrt{n+1}} x^{n+1}}{\frac{(-1)^n 4^n}{\sqrt{n}} x^n} \right| = \lim_{n\to\infty} \left| \frac{4\sqrt{n}}{\sqrt{n+1}} x \right| = 4|x|$$

So $4|x| < 1$ giving $|x| < 1/4$, hence the radius of convergence is $1/4$. Checking the endpoints of $x = 1/4$ and $x = -1/4$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}} \left(\frac{1}{4}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

which converges by the alternating series test.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}} \left(-\frac{1}{4}\right)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

which diverges by the p-series test, $p = 1/2 \leq 1$. So the interval of convergence is $(-1/4, 1/4]$.

3. (*15 Points*): Find a power series representation for the function $f(x) = \dfrac{x^2}{x^4+16}$ and determine the radius of convergence.

Solution:

$$\frac{x^2}{x^4+16} = x^2\frac{1}{16+x^4} = \frac{x^2}{16}\cdot\frac{1}{1+(x/2)^4} = \frac{x^2}{16}\cdot\frac{1}{1-(-(x/2)^4)}$$
$$= \frac{x^2}{16}\sum_{n=0}^{\infty}(-(x/2)^4)^n = \frac{x^2}{16}\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{2^{4n}} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2^{4n+4}}$$

The radius of convergence is when $(x/2)^4 < 1$, that is $x < 2$, so the radius of convergence is 2.

4. (*20 Points*): Find the Taylor series for $f(x) = \ln(x)$ centered at $a = 2$.

Solution:

$$\begin{aligned}
f(2) &= \ln(2) \\
f'(x) &= \frac{1}{x} \qquad f'(2) = \frac{1}{2} \\
f''(x) &= -\frac{1}{x^2} \qquad f''(2) = -\frac{1}{4} \\
f'''(x) &= \frac{2}{x^3} \qquad f'''(2) = \frac{1}{4} \\
f^{(4)}(x) &= -\frac{6}{x^4} \qquad f^{(4)}(2) = -\frac{6}{16} \\
f^{(5)}(x) &= \frac{24}{x^5} \qquad f^{(5)}(2) = \frac{24}{32}
\end{aligned}$$

So for $n \geq 1$, $f^{(n)}(2) = (-1)^{n+1}\frac{(n-1)!}{2^n}$. Hence the Taylor Series is

$$\ln(2) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\frac{(n-1)!}{2^n}}{n!}(x-2)^n = \ln(2) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n2^n}(x-2)^n$$

5. **Extra Credit**: (*10 Points*): Find the exact sum of the series.

$$\sum_{n=0}^{\infty}\frac{(-1)^n\pi^n}{3^{2n}(2n)!}$$

Solution:

$$\sum_{n=0}^{\infty}\frac{(-1)^n\pi^n}{3^{2n}(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n(\sqrt{\pi})^{2n}}{3^{2n}(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\sqrt{\pi}}{3}\right)^{2n}}{(2n)!} = \cos\left(\frac{\sqrt{\pi}}{3}\right) \approx 0.8304853255$$