Fall 2012

Exam #1

Math 306

Name: ____

Write all of your responses on these exam pages. If you need more space please use the backs. Make sure that you show all of your work.

1. **Definitions:** (*3 Points Each*) Give a definition for each of the following.

   (a) A Homogeneous Linear System

   (b) A Consistent Linear System

   (c) The Reduced Row Echelon form of a Matrix

(d) A Linear Combination of a Set of Vectors

(e) The Span of a Set of Vectors

(f) A Linear Transformation

(g) Linear Independence and Dependence

2. **True and False:** (*3 Points Each*) Mark each of the following as either true or false. If the statement is false either give a counterexample or correct the statement so that it is true. The insertion of the word *not* or changing an $=$ to $\neq$ is insufficient for correcting a statement.

   (a) ________ A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ that is not one-to-one must send an infinite number of non-zero vectors in $\mathbb{R}^n$ to the zero vector in $\mathbb{R}^m$.

   (b) ________ Every elementary row operation is reversible using the same type of row operation.

   (c) ________ Any subset of a linearly independent set is linearly independent.

   (d) ________ If a linear system has a 3×5 coefficient matrix with three pivot columns then the solution set to the system can be viewed geometrically as a plane through the origin.

   (e) ________ A solution to a linear system involving variables $x_1, \ldots, x_n$ is a list of numbers $(s_1, \ldots, s_n)$ that makes each equation in the system a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$ respectively.

   (f) ________ If two linear systems have the same set of solutions then the systems are row equivalent.

   (g) ________ $A\mathbf{x}$ is a linear combination of the rows of A.

   (h) ________ Let $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{b}\} \subset \mathbb{R}^n$, then if $\mathbf{b} \in \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ the system $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4\ \mathbf{b}]$ has a solution for any vector $\mathbf{a}_4$.

   (i) ________ If A is an $m \times n$ matrix whose columns span $\mathbb{R}^m$ then for each vector $\mathbf{b} \in \mathbb{R}^m$ the system $A\mathbf{x} = \mathbf{b}$ has an infinite number of solutions.

   (j) ________ Every linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be written as a matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ where A is a unique $m \times n$ matrix with real entries.

3. **Calculations:** Do each of the following.

(a) (*10 Points*) Write the following system as both a matrix equation and as a vector equation. Then solve the system and put your final answer in parametric vector form. Show each step in the reduction and label the reduction step as we did in class. Finally, write the vector $(1, -3, 8)$ as a linear combination of the columns of the coefficient matrix (if possible) and describe the solution set geometrically, if there is one. Keep all numbers in exact form, no approximations.

$$\begin{aligned} -x_1 + 6x_2 + 16x_3 &= 1 \\ 11x_1 - 22x_2 - 44x_3 &= -3 \\ -22x_1 + 55x_2 + 121x_3 &= 8 \end{aligned}$$

(b) (*5 Points*) Consider the transformation T defined by $T(x_1, x_2, x_3) = (2x_1 - x_3 - 1, -x_3 + 2x_2 - x_1 + 3)$.

i. What is the domain of T?

ii. What is the codomain of T?

iii. If A is the matrix such that $T(\mathbf{x}) = A\mathbf{x}$, what is the size of A?

iv. Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

(c) (*5 Points*) Let T be the transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that first rotates the plane counter clockwise about the origin by $\pi/4$ radians, then reflects over the y-axis (the vertical axis) and finally scales by a factor of 2 in the x direction (horizontal) and by 4 in the y direction (vertical). Keep all numbers in exact form, no approximations.

i. Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

ii. Is T a one-to-one map? Justify your answer.

iii. Is T an onto map? Justify your answer.

(d) (*10 Points*) For each of the following sets of vectors, tell me if the set is linearly independent or dependent and why. Do as few operations as possible to answer the question. Also, describe geometrically the span of the set of vectors.

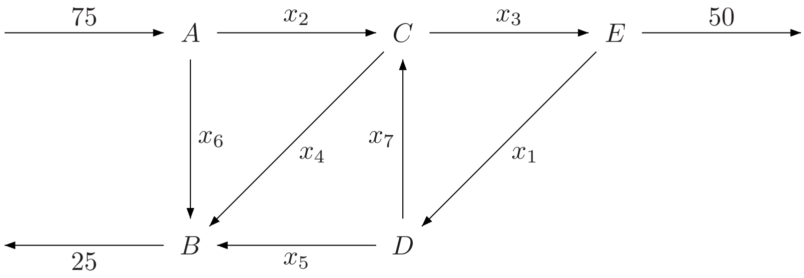
i. $\{(1, -2), (3, -1)\}$

ii. $\{(-2, 3, -1), (1, -3, 2)\}$

iii. $\{(1, 1, 1), (2, 1, -5), (7, 5, -7)\}$

iv. $\{(1, 1, 1), (2, 1, -5), (7, 5, -7), (1, -1, 2)\}$

(e) (*10 Points*) Given the following network flow, set up the matrix that describes the general flow pattern, do not solve the system.



(f) (*10 Points*) In a certain region, about 5% of a city's population moves to the surrounding suburbs each year, and about 4% of the suburban population moves into the city. Currently there are 1,000,000 residents in the city and 750,000 in the suburbs. Set up the migration matrix for this situation and estimate the populations of the city and suburbs two years from now.

(g) (*10 Points*) Find the equation of the parabola that passes through the points $(1, 2)$, $(2, -3)$, and $(3, 4)$.