Fall 2012

Exam #1 Key

Math 306

1. **Definitions:** (*3 Points Each*) Give a definition for each of the following.

   (a) A Homogeneous Linear System — A Homogeneous Linear System is one of the form $A\mathbf{x} = \mathbf{0}$.

   (b) A Consistent Linear System — A Consistent Linear System is one that has at least one solution.

   (c) The Reduced Row Echelon form of a Matrix — The Reduced Row Echelon form of a Matrix has the following properties.

      i. All nonzero rows are above any rows of all zeros.

      ii. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

      iii. All leading entries are 1.

      iv. Each leading 1 is the only nonzero entry in its column.

   (d) A Linear Combination of a Set of Vectors — If the set of vectors is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots \mathbf{v}_n\}$ then a linear combination of these vectors is $\mathbf{w} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + \cdots + x_n\mathbf{v}_n$ where $x_1, x_2, \dots, x_n$ are real number constants called weights.

   (e) The Span of a Set of Vectors — The span is the set of all linear combinations of the vectors in the set.

   (f) A Linear Transformation — A linear transformation T is a function such that the following two conditions hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ and scalar c,

      i. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

      ii. $T(c\mathbf{v}) = cT(\mathbf{v})$

   (g) Linear Independence and Dependence — A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots \mathbf{v}_n\}$ is said to be linearly independent if the vector equation, $\mathbf{0} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + \cdots + x_n\mathbf{v}_n$ has only the trivial solution for the set of weights, that is, $x_1 = x_2 = \cdots = x_n = 0$. If there is a nontrivial solution then the set is dependent.

2. **True and False:** (*3 Points Each*) Mark each of the following as either true or false. If the statement is false either give a counterexample or correct the statement so that it is true. The insertion of the word *not* or changing an $=$ to $\neq$ is insufficient for correcting a statement.

   (a) **TRUE:** A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ that is not one-to-one must send an infinite number of non-zero vectors in $\mathbb{R}^n$ to the zero vector in $\mathbb{R}^m$.

   (b) **TRUE:** Every elementary row operation is reversible using the same type of row operation.

   (c) **TRUE:** Any subset of a linearly independent set is linearly independent.

   (d) **FALSE:** If a linear system has a 3×5 coefficient matrix with three pivot columns then the solution set to the system can be viewed geometrically as a plane through the origin. — The solution set will be a plane but not necessarily through the origin.

   (e) **TRUE:** A solution to a linear system involving variables $x_1, \dots, x_n$ is a list of numbers $(s_1, \dots, s_n)$ that makes each equation in the system a true statement when the values $s_1, \dots, s_n$ are substituted for $x_1, \dots, x_n$ respectively.

   (f) **FALSE:** If two linear systems have the same set of solutions then the systems are row equivalent. — Two systems can have the same solution set but a different number of equations, hence they cannot be row equivalent.

   (g) **FALSE:** $A\mathbf{x}$ is a linear combination of the rows of A. — The columns of A,

   (h) **TRUE:** Let $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{b}\} \subset \mathbb{R}^n$, then if $\mathbf{b} \in \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ the system $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4\ \mathbf{b}]$ has a solution for any vector $\mathbf{a}_4$.

   (i) **FALSE:** If A is an $m \times n$ matrix whose columns span $\mathbb{R}^m$ then for each vector $\mathbf{b} \in \mathbb{R}^m$ the system $A\mathbf{x} = \mathbf{b}$ has an infinite number of solutions. — The system always has a solution but it may be a unique solution.

   (j) **TRUE:** Every linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be written as a matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ where A is a unique $m \times n$ matrix with real entries.

3. **Calculations:** Do each of the following.

(a) (*10 Points*) Write the following system as both a matrix equation and as a vector equation. Then solve the system and put your final answer in parametric vector form. Show each step in the reduction and label the reduction step as we did in class. Finally, write the vector $(1, -3, 8)$ as a linear combination of the columns of the coefficient matrix (if possible) and describe the solution set geometrically, if there is one. Keep all numbers in exact form, no approximations.

$$\begin{aligned} -x_1 + 6x_2 + 16x_3 &= 1 \\ 11x_1 - 22x_2 - 44x_3 &= -3 \\ -22x_1 + 55x_2 + 121x_3 &= 8 \end{aligned}$$

Solution: Matrix and Vector Equations

$$\begin{bmatrix} -1 & 6 & 16 \\ 11 & -22 & -44 \\ -22 & 55 & 121 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 8 \end{bmatrix} \qquad x_1 \begin{bmatrix} -1 \\ 11 \\ -22 \end{bmatrix} + x_2 \begin{bmatrix} 6 \\ -22 \\ 55 \end{bmatrix} + x_3 \begin{bmatrix} 16 \\ -44 \\ 121 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 8 \end{bmatrix}$$

Solving

$$\begin{bmatrix} -1 & 6 & 16 & 1 \\ 11 & -22 & -44 & -3 \\ -22 & 55 & 121 & 8 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & -6 & -16 & -1 \\ 11 & -22 & -44 & -3 \\ -22 & 55 & 121 & 8 \end{bmatrix} \xrightarrow{-11R_1+R_2}$$

$$\begin{bmatrix} 1 & -6 & -16 & -1 \\ 0 & 44 & 132 & 8 \\ -22 & 55 & 121 & 8 \end{bmatrix} \xrightarrow{22R_1+R_3} \begin{bmatrix} 1 & -6 & -16 & -1 \\ 0 & 44 & 132 & 8 \\ 0 & -77 & -231 & -14 \end{bmatrix} \xrightarrow{\frac{1}{44}R_2}$$

$$\begin{bmatrix} 1 & -6 & -16 & -1 \\ 0 & 1 & 3 & \frac{2}{11} \\ 0 & -77 & -231 & -14 \end{bmatrix} \xrightarrow{77R_2+R_3} \begin{bmatrix} 1 & -6 & -16 & -1 \\ 0 & 1 & 3 & \frac{2}{11} \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{6R_2+R_1} \begin{bmatrix} 1 & 0 & 2 & \frac{1}{11} \\ 0 & 1 & 3 & \frac{2}{11} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Parametric Vector Form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{11} - 2x_3 \\ \frac{2}{11} - 3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{11} \\ \frac{2}{11} \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} x_3$$

Linear Combination

$$\frac{1}{11} \begin{bmatrix} -1 \\ 11 \\ -22 \end{bmatrix} + \frac{2}{11} \begin{bmatrix} 6 \\ -22 \\ 55 \end{bmatrix} + 0 \begin{bmatrix} 16 \\ -44 \\ 121 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 8 \end{bmatrix}$$

Geometric Interpretation: The set of solutions is a line through the point $\left(\frac{1}{11}, \frac{2}{11}, 0\right)$ and parallel to the vector $(-2, -3, 1)$.

(b) (*5 Points*) Consider the transformation T defined by $T(x_1, x_2, x_3) = (2x_1 - x_3 - 1, -x_3 + 2x_2 - x_1 + 3)$.

i. What is the domain of T? — $\mathbb{R}^3$

ii. What is the codomain of T? — $\mathbb{R}^2$

iii. If A is the matrix such that $T(\mathbf{x}) = A\mathbf{x}$, what is the size of A? — T is not linear so there is no matrix A. Note that $T(\mathbf{0}) = (-1, 3) \neq \mathbf{0}$.

iv. Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$. — T is not linear so there is no matrix A.

(c) (*5 Points*) Let T be the transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that first rotates the plane counter clockwise about the origin by $\pi/4$ radians, then reflects over the y-axis (the vertical axis) and finally scales by a factor of 2 in the x direction (horizontal) and by 4 in the y direction (vertical). Keep all numbers in exact form, no approximations.

i. Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.
Solution: $T(\mathbf{e}_1) = (-\sqrt{2}, 2\sqrt{2})$ and $T(\mathbf{e}_2) = (\sqrt{2}, 2\sqrt{2})$ so the matrix is

$$A = \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 2\sqrt{2} & 2\sqrt{2} \end{bmatrix}$$

ii. Is T a one-to-one map? Justify your answer.
Solution: Reducing the matrix A,

$$\begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 2\sqrt{2} & 2\sqrt{2} \end{bmatrix} \xrightarrow{2R_1+R_2} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 0 & 4\sqrt{2} \end{bmatrix}$$

shows that there is a pivot in each column and hence the map is one-to-one.

iii. Is T an onto map? Justify your answer.
Solution: By the above reduction there is a pivot in each row and hence the map is onto.

(d) (*10 Points*) For each of the following sets of vectors, tell me if the set is linearly independent or dependent and why. Do as few operations as possible to answer the question. Also, describe geometrically the span of the set of vectors.

i. $\{(1, -2), (3, -1)\}$ — Independent since there are two vectors that are not scalar multiples of each other. Their span is all of $\mathbb{R}^2$.

ii. $\{(-2, 3, -1), (1, -3, 2)\}$ — Independent since there are two vectors that are not scalar multiples of each other. Their span is a plane through the origin in $\mathbb{R}^3$.

iii. $\{(1, 1, 1), (2, 1, -5), (7, 5, -7)\}$ — Reducing the matrix with the vectors as columns we have,

$$\begin{bmatrix} 1 & 2 & 7 \\ 1 & 1 & 5 \\ 1 & -5 & -7 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 2 & 7 \\ 0 & -1 & -2 \\ 1 & -5 & -7 \end{bmatrix} \xrightarrow{-R_1+R_3} \begin{bmatrix} 1 & 2 & 7 \\ 0 & -1 & -2 \\ 0 & -7 & -14 \end{bmatrix}$$

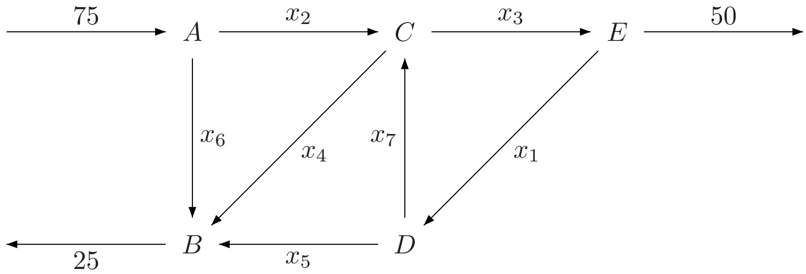
At this point we see that rows 2 and 3 are scalar multiples of each other, indicating that there is a non-trivial solution to the homogeneous system and hence the set of vectors is dependent. Furthermore, the third vector is a linear combination of the first two. Since there are pivots in columns 1 and 2 these two vectors form an independent set, so the span will be a plane through the origin in $\mathbb{R}^3$.

iv. $\{(1, 1, 1), (2, 1, -5), (7, 5, -7), (1, -1, 2)\}$ — From the above exercise we know that the third vector is a linear combination of the first two and hence this is a dependent set. One can also note that we have 4 vectors in $\mathbb{R}^3$, hence the set is dependent. For the span one needs to consider vectors 1, 2, and 4. Reducing the matrix with the vectors as columns we have,

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 1 & -5 & 2 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 1 & -5 & 2 \end{bmatrix} \xrightarrow{-R_1+R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & -7 & 1 \end{bmatrix}$$

At this point we see that rows 2 and 3 are not scalar multiples of each other, indicating that there is only the trivial solution to the homogeneous system and hence this set of vectors is independent. Since there are pivots in every row the span will be all of $\mathbb{R}^3$.

(e) (*10 Points*) Given the following network flow, set up the matrix that describes the general flow pattern, do not solve the system.



Solution:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 75 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 25 \\ 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 50 \end{bmatrix}$$

(f) (*10 Points*) In a certain region, about 5% of a city's population moves to the surrounding suburbs each year, and about 4% of the suburban population moves into the city. Currently there are 1,000,000 residents in the city and 750,000 in the suburbs. Set up the migration matrix for this situation and estimate the populations of the city and suburbs two years from now.

Solution: The migration matrix is

$$M = \begin{bmatrix} 0.95 & 0.04 \\ 0.05 & 0.96 \end{bmatrix}$$

In one year the populations are estimated to be,

$$\begin{bmatrix} 0.95 & 0.04 \\ 0.05 & 0.96 \end{bmatrix} \begin{bmatrix} 1000000 \\ 750000 \end{bmatrix} = \begin{bmatrix} 980000 \\ 770000 \end{bmatrix}$$

After two years the populations are estimated to be,

$$\begin{bmatrix} 0.95 & 0.04 \\ 0.05 & 0.96 \end{bmatrix} \begin{bmatrix} 980000 \\ 770000 \end{bmatrix} = \begin{bmatrix} 961800 \\ 788200 \end{bmatrix}$$

(g) (*10 Points*) Find the equation of the parabola that passes through the points $(1, 2)$, $(2, -3)$, and $(3, 4)$.

Solution:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & -3 \\ 1 & 3 & 9 & 4 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -5 \\ 1 & 3 & 9 & 4 \end{bmatrix} \xrightarrow{-R_1+R_3} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 2 & 8 & 2 \end{bmatrix} \xrightarrow{-2R_2+R_3}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 2 & 12 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & 6 \end{bmatrix} \xrightarrow{-3R_3+R_2} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -23 \\ 0 & 0 & 1 & 6 \end{bmatrix} \xrightarrow{-R_3+R_1}$$

$$\begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 1 & 0 & -23 \\ 0 & 0 & 1 & 6 \end{bmatrix} \xrightarrow{-R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & 19 \\ 0 & 1 & 0 & -23 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

So the quadratic is

$$y = 6x^2 - 23x + 19$$