Spring 2012

Exam #2

Math 306

Name: ______________________________

Write all of your responses on these exam pages. If you need more space please use the backs. Make sure that you show all of your work.

1. **Definitions and Short Answer:** (*3 Points Each*) Give a definition or short answer for each of the following.

   (a) A Subspace of $\mathbb{R}^n$

   (b) A Basis for a subspace of $\mathbb{R}^n$

   (c) The Rank of a matrix

   (d) State the Rank Theorem

   (e) State the Basis Theorem

2. **True and False:** (*3 Points Each*) Mark each of the following as either true or false. If the statement is false either give a counterexample or explain why the statement is false.

   (a) ______ If the columns of two square matrices A and B, of the same size, are independent then $A + B$ is an invertible matrix.

   (b) ______ The determinant of an echelon form of a matrix is ± 1 times the determinant of the original matrix.

   (c) ______ If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^n$ is a linearly independent set then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ forms a basis for $\mathbb{R}^n$.

   (d) ______ If A is a square matrix such that $A^T A = I$ then $\det(A) = 1$.

   (e) ______ For $n \times n$ matrices A and B, $A^2 - B^2 = (A + B)(A - B)$.

   (f) ______ If $AB = 0$ for non-zero $n \times n$ matrices A and B, then neither A nor B can be invertible.

   (g) ______ If A is an $n \times m$ matrix such that the columns of A span $\mathbb{R}^n$ then the columns of A are linearly independent and form a basis for $\mathbb{R}^n$.

   (h) ______ If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subset \mathbb{R}^n$ with $m > n$ then $\text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\})$ is a subspace of $\mathbb{R}^n$.

   (i) ______ If T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $T(\mathbf{x}) = A\mathbf{x}$ and the Rank of A is n then T is a one-to-one map.

   (j) ______ If B is the reduced echelon form of the matrix A then $B = MA$ for some invertible matrix M, furthermore, $|A| = \frac{1}{|M|}$.

3. **Calculations:** (*10 Points Each*) Do each of the following.

(a) Given that

$$A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 0 \\ -2 & 4 \\ 1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 2 & -3 \end{bmatrix}$$

find the following if they exist. If the operation is not defined state why?

i. $3A - 2C$

ii. AB

iii. AC

iv. ABC

v. $A - B^T$

(b) Find the inverse of the following matrix, A. Show all of the steps in the derivations and keep your answers in exact form.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 2 & 2 \\ 1 & -3 & 0 \end{bmatrix}$$

(c) Find the determinant of the following matrix using cofactor expansion. You may use a short-cut method when taking the determinant of a 2×2 matrix but none larger. Show all of the steps in the derivations and keep your answers in exact form.

$$A = \begin{bmatrix} -8 & 0 & -4 & 0 \\ -6 & -8 & -3 & 1 \\ 10 & 7 & 0 & -3 \\ 7 & 0 & -9 & 1 \end{bmatrix}$$

(d) Find the determinant of the following matrix using the reduction method. You may not use a short-cut method nor a combination of reduction and cofactor expansion. The reduction method must be carried out to at least echelon form. Show all of the steps in the derivations and keep your answers in exact form.

$$A = \begin{bmatrix} -8 & 0 & -4 & 0 \\ -6 & -8 & -3 & 1 \\ 10 & 7 & 0 & -3 \\ 7 & 0 & -9 & 1 \end{bmatrix}$$

(e) For the following matrix A find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$. Show all of the steps in the derivations and keep your answers in exact form.

$$A = \begin{bmatrix} -1 & 1 & -1 & 1 & 0 \\ 2 & -1 & 4 & -1 & 4 \\ -5 & 2 & -11 & 1 & -17 \end{bmatrix}$$

4. **Proofs:** (*5 Points Each*) Do each of the following.

   (a) Suppose that A is an $m \times n$ matrix and D is an $n \times m$ matrix with $AD = I_m$. Show that for any $\mathbf{b} \in \mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

   (b) Two square matrices A and B are said to be similar if there exists an invertible matrix C with $A = CBC^{-1}$. Show that similar matrices have the same determinant.

(c) Let A be an $n \times n$ singular matrix. Describe how to construct a nonzero $n \times n$ matrix B such that $AB = 0$.