Spring 2012

Exam #2 Key

Math 306

1. **Definitions and Short Answer:** (*3 Points Each*) Give a definition or short answer for each of the following.

   (a) A Subspace of $\mathbb{R}^n$: A Subspace of $\mathbb{R}^n$ is a subset H with the following properties,

      i. $\mathbf{0} \in H$.

      ii. For all vectors $\mathbf{u}$ and $\mathbf{v}$ in H their sum $\mathbf{u} + \mathbf{v}$ is in H.

      iii. For all vectors $\mathbf{u}$ in H and scalars c the vector $c\mathbf{u}$ is in H.

   (b) A Basis for a subspace of $\mathbb{R}^n$: A Basis for a subspace H of $\mathbb{R}^n$ is a set of vectors from H that are both linearly independent and span H.

   (c) The Rank of a matrix: The Rank of a matrix is the dimension of the column space of the matrix.

   (d) State the Rank Theorem: For any $m \times n$ matrix A

   $$\operatorname{rank}(A) + \dim(\operatorname{Nul}(A)) = n$$

   (e) State the Basis Theorem: Let H is a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of exactly p elements in H that spans H is automatically a basis for H.

2. **True and False:** (*3 Points Each*) Mark each of the following as either true or false. If the statement is false either give a counterexample or explain why the statement is false.

   (a) **FALSE:** If the columns of two square matrices A and B, of the same size, are independent then $A + B$ is an invertible matrix. — If $A = -B$ then they could both have columns forming independent sets but the sum would be the zero matrix and hence not invertible.

   (b) **FALSE:** The determinant of an echelon form of a matrix is ± 1 times the determinant of the original matrix. — Only if no row scaling operations were done in the reduction process.

   (c) **TRUE:** If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^n$ is a linearly independent set then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ forms a basis for $\mathbb{R}^n$.

   (d) **FALSE:** If A is a square matrix such that $A^T A = I$ then $\det(A) = 1$. — $\det(A) = \pm 1$

   (e) **FALSE:** For $n \times n$ matrices A and B, $A^2 - B^2 = (A+B)(A-B)$. — $(A+B)(A-B) = A^2 + BA - AB + B^2$ and this is only equal to $A^2 - B^2$ if A and B commute, which in general is not the case.

   (f) **TRUE:** If $AB = 0$ for non-zero $n \times n$ matrices A and B, then neither A nor B can be invertible.

   (g) **FALSE:** If A is an $n \times m$ matrix such that the columns of A span $\mathbb{R}^n$ then the columns of A are linearly independent and form a basis for $\mathbb{R}^n$. — Only if $n = m$.

   (h) **TRUE:** If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subset \mathbb{R}^n$ with $m > n$ then $\mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\})$ is a subspace of $\mathbb{R}^n$.

   (i) **TRUE:** If T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $T(\mathbf{x}) = A\mathbf{x}$ and the Rank of A is n then T is a one-to-one map.

   (j) **FALSE:** If B is the reduced echelon form of the matrix A then $B = MA$ for some invertible matrix M, furthermore, $|A| = \frac{1}{|M|}$. — $B = MA$ for some invertible matrix M but it is not necessarily true that $|A| = \frac{1}{|M|}$. Consider the case where $|B| = 0$, since M is invertible $|A| = 0$ as well and clearly $|A| \neq \frac{1}{|M|}$.

3. **Calculations:** (*10 Points Each*) Do each of the following.

(a) Given that

$$A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 0 \\ -2 & 4 \\ 1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 2 & -3 \end{bmatrix}$$

find the following if they exist. If the operation is not defined state why?

i. $3A - 2C = \begin{bmatrix} -2 & 11 & -10 \\ -9 & 8 & 21 \end{bmatrix}$

ii. $AB = \begin{bmatrix} -2 & 10 \\ -6 & 21 \end{bmatrix}$

iii. AC does not exist since the number of columns of A does not match the number of rows of C.

iv. $ABC = \begin{bmatrix} 22 & 22 & -34 \\ 39 & 48 & -75 \end{bmatrix}$

v. $A - B^T = \begin{bmatrix} -1 & 5 & -3 \\ -1 & 0 & 4 \end{bmatrix}$

(b) Find the inverse of the following matrix, A. Show all of the steps in the derivations and keep your answers in exact form.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 2 & 2 \\ 1 & -3 & 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & -3 & 0 & 0 & 0 & 1 \\ -2 & 2 & 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 0 & 0 & 0 & 1 \\ 0 & -4 & 2 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & -3 & 0 & 0 & 0 & 1 \\ 0 & -4 & 2 & 0 & 1 & 2 \\ 0 & 7 & 0 & 1 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 0 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 7 & 0 & 1 & 0 & -2 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & -3 & 0 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & \frac{7}{2} & 1 & \frac{7}{4} & \frac{3}{2} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 0 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{2}{7} & \frac{1}{2} & \frac{3}{7} \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & -3 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{7} & 0 & -\frac{2}{7} \\ 0 & 0 & 1 & \frac{2}{7} & \frac{1}{2} & \frac{3}{7} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{3}{7} & 0 & \frac{1}{7} \\ 0 & 1 & 0 & \frac{1}{7} & 0 & -\frac{2}{7} \\ 0 & 0 & 1 & \frac{2}{7} & \frac{1}{2} & \frac{3}{7} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{3}{7} & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & -\frac{2}{7} \\ \frac{2}{7} & \frac{1}{2} & \frac{3}{7} \end{bmatrix}$$

(c) Find the determinant of the following matrix using cofactor expansion. You may use a short-cut method when taking the determinant of a 2×2 matrix but none larger. Show all of the steps in the derivations and keep your answers in exact form.

$$A = \begin{bmatrix} -8 & 0 & -4 & 0 \\ -6 & -8 & -3 & 1 \\ 10 & 7 & 0 & -3 \\ 7 & 0 & -9 & 1 \end{bmatrix}$$

Solution:

$$\begin{aligned}
|A| &= -8 \cdot \begin{vmatrix} -8 & -4 & 0 \\ 10 & 0 & -3 \\ 7 & -9 & 1 \end{vmatrix} - 7 \cdot \begin{vmatrix} -8 & -4 & 0 \\ -6 & -3 & 1 \\ 7 & -9 & 1 \end{vmatrix} \\
&= -8 \left(3 \cdot \begin{vmatrix} -8 & -4 \\ 7 & -9 \end{vmatrix} + \begin{vmatrix} -8 & -4 \\ 10 & 0 \end{vmatrix} \right) - 7 \left(-1 \cdot \begin{vmatrix} -8 & -4 \\ 7 & -9 \end{vmatrix} + \begin{vmatrix} -8 & -4 \\ -6 & -3 \end{vmatrix} \right) \\
&= -8\,(3 \cdot 100 + 40) - 7\,(-1 \cdot 100 + 0) = -2020
\end{aligned}$$

(d) Find the determinant of the following matrix using the reduction method. You may not use a short-cut method nor a combination of reduction and cofactor expansion. The reduction method must be carried out to at least echelon form. Show all of the steps in the derivations and keep your answers in exact form.

$$A = \begin{bmatrix} -8 & 0 & -4 & 0 \\ -6 & -8 & -3 & 1 \\ 10 & 7 & 0 & -3 \\ 7 & 0 & -9 & 1 \end{bmatrix}$$

Solution:

$$\begin{vmatrix} -8 & 0 & -4 & 0 \\ -6 & -8 & -3 & 1 \\ 10 & 7 & 0 & -3 \\ 7 & 0 & -9 & 1 \end{vmatrix} = \begin{vmatrix} -8 & 0 & -4 & 0 \\ 1 & -8 & -12 & 2 \\ 10 & 7 & 0 & -3 \\ 7 & 0 & -9 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & -8 & -12 & 2 \\ -8 & 0 & -4 & 0 \\ 10 & 7 & 0 & -3 \\ 7 & 0 & -9 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & -8 & -12 & 2 \\ 0 & -64 & -100 & 16 \\ 0 & 87 & 120 & -23 \\ 0 & 56 & 75 & -13 \end{vmatrix} = 64 \begin{vmatrix} 1 & -8 & -12 & 2 \\ 0 & 1 & \frac{25}{16} & -\frac{1}{4} \\ 0 & 87 & 120 & -23 \\ 0 & 56 & 75 & -13 \end{vmatrix} = 64 \begin{vmatrix} 1 & -8 & -12 & 2 \\ 0 & 1 & \frac{25}{16} & -\frac{1}{4} \\ 0 & 0 & -\frac{255}{16} & -\frac{5}{4} \\ 0 & 0 & -\frac{25}{2} & 1 \end{vmatrix}$$

$$= 64 \cdot -\frac{255}{16} \begin{vmatrix} 1 & -8 & -12 & 2 \\ 0 & 1 & \frac{25}{16} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{4}{51} \\ 0 & 0 & -\frac{25}{2} & 1 \end{vmatrix} = 64 \cdot -\frac{255}{16} \begin{vmatrix} 1 & -8 & -12 & 2 \\ 0 & 1 & \frac{25}{16} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{4}{51} \\ 0 & 0 & 0 & \frac{101}{51} \end{vmatrix}$$

$$= 64 \cdot -\frac{255}{16} \cdot \frac{101}{51} = -2020$$

(e) For the following matrix A find bases for Nul(A) and Col(A). Show all of the steps in the derivations and keep your answers in exact form.

$$A = \begin{bmatrix} -1 & 1 & -1 & 1 & 0 \\ 2 & -1 & 4 & -1 & 4 \\ -5 & 2 & -11 & 1 & -17 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} -1 & 1 & -1 & 1 & 0 \\ 2 & -1 & 4 & -1 & 4 \\ -5 & 2 & -11 & 1 & -17 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

so a basis for Col(A) would be columns 1, 2, and 4, specifically,

$$\mathcal{B} = \begin{bmatrix} -1 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

For the Nul(A), we write the solution to $A\mathbf{x} = \mathbf{0}$ in parametric vector form,

$$\mathbf{x} = \begin{bmatrix} -3x_3 - 4x_5 \\ -2x_3 + x_5 \\ x_3 \\ -5x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 1 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

so a basis for Nul(A) is

$$\mathcal{B} = \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

4. **Proofs:** (*5 Points Each*) Do each of the following.

(a) Suppose that A is an $m \times n$ matrix and D is an $n \times m$ matrix with $AD = I_m$. Show that for any $\mathbf{b} \in \mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

Solution: Since $I_m\mathbf{b} = \mathbf{b}$ we have $AD\mathbf{b} = \mathbf{b}$, and hence $A(D\mathbf{b}) = \mathbf{b}$. So let $\mathbf{x} = D\mathbf{b}$ and the above simplifies to $A\mathbf{x} = \mathbf{b}$.

(b) Two square matrices A and B are said to be similar if there exists an invertible matrix C with $A = CBC^{-1}$. Show that similar matrices have the same determinant.

Solution: $|A| = |CBC^{-1}| = |C||B||C^{-1}| = |C||B||C|^{-1} = |B|$.

(c) Let A be an $n \times n$ singular matrix. Describe how to construct a nonzero $n \times n$ matrix B such that $AB = 0$.

Solution: Since A is singular, i.e. not invertible, we know that the Null Space of A is not trivial. So take vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots \mathbf{b}_n$ not all $\mathbf{0}$ in the null space of A and construct $B = [\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_n]$. Then B is nonzero and since the columns of B are vectors from the null space of A we have,

$$AB = A[\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_n] = [A\mathbf{b}_1\ A\mathbf{b}_2\ \cdots\ A\mathbf{b}_n] = 0$$