Fall 2012 **Exam #3** Math 306

Exam #3

Name: ______________________________

Write all of your responses on these exam pages. If you need more space please use the backs. Make sure that you show all of your work.

1. **Definitions and Short Answer:** (*3 Points Each*) Give a definition or short answer for each of the following.

   (a) A Subspace of a vector space V.

   (b) The Kernel of a Linear Transformation.

   (c) An Eigenvalue and Eigenvector of a matrix A.

   (d) The Coordinate Vector of a vector $\mathbf{x}$ in a vector space V with respect to a given ordered basis $\mathcal{B}$.

   (e) The Characteristic Equation of a matrix A.

2. **True and False:** (*3 Points Each*) Mark each of the following as either true or false. If the statement is false either give a counterexample or explain why the statement is false.

   (a) ______ $\mathbb{R}^2$ is a subspace of $\mathbb{R}^4$.

   (b) ______ If $\mathbf{f}$ is a vector in the vector space $C[0,1]$ of all continuous functions on the interval $[0,1]$ and $f(t) = 0$ for some $t \in [0,1]$ then $\mathbf{f}$ is the zero vector of the vector space.

   (c) ______ Given a linear transformation $T : V \to W$, the range of T is a subspace of V.

   (d) ______ A set consisting of a single vector forms a linearly independent set.

   (e) ______ Given any linearly independent set of vectors S in a finite dimensional vector space V there is a subset of S that forms a basis to V.

   (f) ______ If $P_{\mathcal{B}}$ is the change of coordinate matrix, then for any vector $\mathbf{x} \in V$ we have $P_{\mathcal{B}}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$.

   (g) ______ The eigenvalues of a matrix are on the main diagonal in the reduced echelon form of the matrix.

   (h) ______ Given an $n \times n$ matrix $A \neq I_n$ with real entries, such that $A^4 = I_n$, then the only eigenvalue of A is 1.

   (i) ______ If a matrix A is invertible then A is diagonalizable.

   (j) ______ If $AP = PD$ with D diagonal then the columns of P are eigenvectors of A.

3. **Calculations:** Do each of the following. Keep all of your answers in exact form.

   (a) (*10 Points*) Let $S = \{1, \sin(x), \sin^2(x), \cos(2x)\}$ and let $H = \text{Span}(S)$. Show that the set S is linearly dependent and then extract a subset of S that is a basis for H. Verify that it is a basis.

(b) (*10 Points*) Given the basis $\mathcal{B} = \{(1,-1,2),(3,2,-1),(2,4,-3)\}$ of $\mathbb{R}^3$ find $P_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{B}}$ where $\mathbf{x} = (1,1,1)$.

(c) (*10 Points*) Given the two ordered bases $\mathcal{B} = \{2, 5 + x, 3 - 2x + x^2\}$ and $\mathcal{C} = \{1 - x, 3 + 2x, 1 + x^2\}$ of $\mathbb{P}_2$. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be defined as differentiation, that is, $T(ax^2 + bx + c) = 2ax + b$. Find the matrix of T relative to the bases $\mathcal{B}$ and $\mathcal{C}$.

(d) (*20 Points*) Consider the following matrix A.

$$A = \begin{bmatrix} 3 & 3 & -3 \\ -4 & -5 & 6 \\ -2 & -3 & 4 \end{bmatrix}$$

i. Find the characteristic polynomial of the matrix A.
ii. Find the eigenvalues of the matrix A.
iii. What is the algebraic multiplicity of each eigenvalue?
iv. For each eigenvalue, find a basis to the eigenspace for that eigenvalue.
v. What is the dimension of each eigenspace?
vi. Is the matrix A diagonalizable? If so, find matrices P and D such that D is diagonal and $A = PDP^{-1}$ and if not explain why.

(More space for exercise 3d)

4. **Proofs:** (*5 Points Each*) Do each of the following.

   (a) Prove that if two $n \times n$ matrices are similar then they have the same eigenvalues.

   (b) Show that if A is both diagonalizable and invertible then so is A^{-1}.

(c) Show that the coordinate mapping is both one-to-one and onto.