Fall 2012 **Exam #3 Key** Math 306

Exam #3 Key

1. **Definitions and Short Answer:** (*3 Points Each*) Give a definition or short answer for each of the following.

   (a) A Subspace of a vector space V. — A subset H of a vector space V is a subspace of V if $\mathbf{0} \in H$ and for all vectors $\mathbf{x}$ and $\mathbf{y}$ in H and scalars $c \in \mathbb{R}$, $\mathbf{x} + \mathbf{y} \in H$ and $c\mathbf{x} \in H$.

   (b) The Kernel of a Linear Transformation. — $\ker(T) = \{\mathbf{x} \mid T(\mathbf{x}) = \mathbf{0}\}$.

   (c) An Eigenvalue and Eigenvector of a matrix A. — Let $\mathbf{x}$ be a non-zero vector and $\lambda \in \mathbb{R}$ such that $A\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{x}$ is an eigenvector of A associated with the eigenvalue λ.

   (d) The Coordinate Vector of a vector $\mathbf{x}$ in a vector space V with respect to a given ordered basis $\mathcal{B}$. — Denote the basis $\mathcal{B}$ as $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$, then for any $\mathbf{x} \in V$, $\mathbf{x}$ can be written uniquely as $\mathbf{x} = r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + \cdots + r_n\mathbf{b}_n$. In this notation $[\mathbf{x}]_{\mathcal{B}} = (r_1, r_2, \ldots, r_n)$.

   (e) The Characteristic Equation of a matrix A. — $\det(A - \lambda I) = 0$.

2. **True and False:** (*3 Points Each*) Mark each of the following as either true or false. If the statement is false either give a counterexample or explain why the statement is false.

   (a) **FALSE:** $\mathbb{R}^2$ is a subspace of $\mathbb{R}^4$. — $\mathbb{R}^2 \not\subset \mathbb{R}^4$

   (b) **FALSE:** If $\mathbf{f}$ is a vector in the vector space $C[0,1]$ of all continuous functions on the interval $[0,1]$ and $f(t) = 0$ for some $t \in [0,1]$ then $\mathbf{f}$ is the zero vector of the vector space. — $f(t) = 0$ for all $t \in [0,1]$.

   (c) **FALSE:** Given a linear transformation $T : V \to W$, the range of T is a subspace of V. — The range of T is a subspace of W.

   (d) **FALSE:** A set consisting of a single vector forms a linearly independent set. — Only if the vector is not the zero vector.

   (e) **FALSE:** Given any linearly independent set of vectors S in a finite dimensional vector space V there is a subset of S that forms a basis to V. — If S is a spanning set of V.

   (f) **FALSE:** If $P_{\mathcal{B}}$ is the change of coordinate matrix, then for any vector $\mathbf{x} \in V$ we have $P_{\mathcal{B}}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$. — $P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$

   (g) **FALSE:** The eigenvalues of a matrix are on the main diagonal in the reduced echelon form of the matrix. — Row reduction can change the eigenvalues of a matrix.

   (h) **FALSE:** Given an $n \times n$ matrix $A \neq I_n$ with real entries, such that $A^4 = I_n$, then the only eigenvalue of A is 1. — $\mathbf{x} = I\mathbf{x} = A^4\mathbf{x} = \lambda^4\mathbf{x}$, so $\lambda = \pm 1$, excluding complex numbers.

   (i) **FALSE:** If a matrix A is invertible then A is diagonalizable. — The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible with 1 as the only eigenvalue (hence of multiplicity 2). Forming $A - I$ we see there is only one free variable and hence the dimension of the eigenspace is one and not two, so A is not diagonalizable.

   (j) **FALSE:** If $AP = PD$ with D diagonal then the columns of P are eigenvectors of A. — We also need that P is invertible. Note that if $P = 0$ the equation is still valid and P would contain no eigenvectors of A.

3. **Calculations:** Do each of the following. Keep all of your answers in exact form.

(a) (*10 Points*) Let $S = \{1, \sin(x), \sin^2(x), \cos(2x)\}$ and let $H = \text{Span}(S)$. Show that the set S is linearly dependent and then extract a subset of S that is a basis for H. Verify that it is a basis.

Solution: $\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x)$, so $\cos(2x)$ is a linear combination of the other three vectors and hence S is a dependent set. Removing $\cos(2x)$ from S gives us $S' = \{1, \sin(x), \sin^2(x)\}$. S' is a basis for H since it is still a spanning set and it is now linearly independent. To see the independence, select three values for x, (0, $\pi/2$ and $-\pi/2$). From these we get the following matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

which reduces to the identity, verifying that the set S' is linearly independent and thus forms a basis for H.

(b) (*10 Points*) Given the basis $\mathcal{B} = \{(1, -1, 2), (3, 2, -1), (2, 4, -3)\}$ of $\mathbb{R}^3$ find $P_\mathcal{B}$ and $[\mathbf{x}]_\mathcal{B}$ where $\mathbf{x} = (1, 1, 1)$.

Solution: $P_\mathcal{B}$ is

$$P_\mathcal{B} = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 4 \\ 2 & -1 & -3 \end{bmatrix}$$

to find $[\mathbf{x}]_\mathcal{B}$ we solve the following system

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ -1 & 2 & 4 & 1 \\ 2 & -1 & -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{13}{7} \\ 0 & 1 & 0 & -\frac{8}{7} \\ 0 & 0 & 1 & \frac{9}{7} \end{bmatrix} \qquad \text{so} \qquad [\mathbf{x}]_\mathcal{B} = \begin{bmatrix} \frac{13}{7} \\ -\frac{8}{7} \\ \frac{9}{7} \end{bmatrix}$$

(c) (*10 Points*) Given the two ordered bases $\mathcal{B} = \{2, 5 + x, 3 - 2x + x^2\}$ and $\mathcal{C} = \{1 - x, 3 + 2x, 1 + x^2\}$ of $\mathbb{P}_2$. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be defined as differentiation, that is, $T(ax^2 + bx + c) = 2ax + b$. Find the matrix of T relative to the bases $\mathcal{B}$ and $\mathcal{C}$.

Solution: $T(2) = 0$, $T(5 + x) = 1$, and $T(3 - 2x + x^2) = -2 + 2x$. To construct the matrix

$$M = \left[\ [T(\mathbf{b}_1)]_\mathcal{C}\ [T(\mathbf{b}_2)]_\mathcal{C}\ [T(\mathbf{b}_3)]_\mathcal{C}\ \right]$$

we only need to reduce the system

$$\begin{bmatrix} 1 & 3 & 1 & 0 & 1 & -2 \\ -1 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{2}{5} & -2 \\ 0 & 1 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad \text{so} \qquad M = \begin{bmatrix} 0 & \frac{2}{5} & -2 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(d) (*20 Points*) Consider the following matrix A.

$$A = \begin{bmatrix} 3 & 3 & -3 \\ -4 & -5 & 6 \\ -2 & -3 & 4 \end{bmatrix}$$

i. Find the characteristic polynomial of the matrix A.
Solution: $|A - xI| = -x^3 + 2x^2 - x = -(x-1)^2 x$

ii. Find the eigenvalues of the matrix A.
Solution: $\lambda = 0$ and $\lambda = 1$.

iii. What is the algebraic multiplicity of each eigenvalue?
Solution: $\lambda = 0$ has multiplicity 1 and $\lambda = 1$ has multiplicity 2.

iv. For each eigenvalue, find a basis to the eigenspace for that eigenvalue.
Solution: For $\lambda = 0$,

$$A - 0I = A = \begin{bmatrix} 3 & 3 & -3 \\ -4 & -5 & 6 \\ -2 & -3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

so a basis for the eigenspace for $\lambda = 0$ is $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$. For $\lambda = 1$,

$$A - I = \begin{bmatrix} 2 & 3 & -3 \\ -4 & -6 & 6 \\ -2 & -3 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so a basis for the eigenspace for $\lambda = 1$ is $\left\{ \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \right\}$.

v. What is the dimension of each eigenspace?
Solution: $\dim(E_0) = 1$ and $\dim(E_1) = 2$

vi. Is the matrix A diagonalizable? If so, find matrices P and D such that D is diagonal and $A = PDP^{-1}$ and if not explain why.
Solution: Yes, since the dimension of each eigenspace matches the algebraic multiplicity of the corresponding eigenvalue.

$$P = \begin{bmatrix} -1 & -3 & 3 \\ 2 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. **Proofs:** (*5 Points Each*) Do each of the following.

   (a) Prove that if two $n \times n$ matrices are similar then they have the same eigenvalues.

   Solution: Let $A = PBP^{-1}$, then

$$|A - \lambda I| = |PBP^{-1} - \lambda I| = |PBP^{-1} - \lambda PIP^{-1}| = |P||B - \lambda I||P^{-1}| = |B - \lambda I|$$

   so both A and B have the same characteristic polynomial and hence the same eigenvalues.

   (b) Show that if A is both diagonalizable and invertible then so is A^{-1}.

   Solution: Since A is diagonalizable there exists an invertible matrix P and a diagonal matrix D with $A = PDP^{-1}$. Since A is invertible then D must also be invertible since $D = P^{-1}AP$. Furthermore, the inverse of a diagonal matrix is also a diagonal matrix. Hence

$$A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1}$$

   and so by the definition of diagonalization, A^{-1} is diagonalizable.

   (c) Show that the coordinate mapping is both one-to-one and onto.

   Solution: For notational purposes let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for the vector space V. We first verify one-to-one, suppose that $[\mathbf{u}]_{\mathcal{B}} = (r_1, r_2, \ldots, r_n)$ and $[\mathbf{w}]_{\mathcal{B}} = (s_1, s_2, \ldots, s_n)$, then if $[\mathbf{u}]_{\mathcal{B}} = [\mathbf{w}]_{\mathcal{B}}$ we have that $r_i = s_i$ for all i, so

$$\mathbf{u} = r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + \cdots + r_n\mathbf{b}_n = s_1\mathbf{b}_1 + s_2\mathbf{b}_2 + \cdots + s_n\mathbf{b}_n = \mathbf{w}$$

   therefore, $[\mathbf{u}]_{\mathcal{B}} = [\mathbf{w}]_{\mathcal{B}}$ implies that $\mathbf{u} = \mathbf{w}$ and thus the coordinate map is one-to-one. To verify that the coordinate map is onto take any element $\mathbf{y} \in \mathbb{R}^n$ and denote it as $\mathbf{y} = (r_1, r_2, \ldots, r_n)$. Construct $\mathbf{x} \in V$ as $\mathbf{x} = r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + \cdots + r_n\mathbf{b}_n$. By the definition of the coordinate map

$$[\mathbf{x}]_{\mathcal{B}} = (r_1, r_2, \ldots, r_n) = \mathbf{y}$$

   so given any element from $\mathbf{y} \in \mathbb{R}^n$ there exists an element $\mathbf{x} \in V$ such that $[\mathbf{x}]_{\mathcal{B}} = \mathbf{y}$. This shows that the coordinate map is onto.