Spring 2012

Exam #1 Key

Math 306

1. **Definitions:** (*4 Points Each*) Give a definition for each of the following.

   (a) The Reduced Row Echelon form of a Matrix — The Reduced Row Echelon form of a Matrix has the following properties.

      i. All nonzero rows are above any rows of all zeros.
      ii. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
      iii. All leading entries are 1.
      iv. Each leading 1 is the only nonzero entry in its column.

   (b) A Linear Combination of a Set of Vectors — If the set of vectors is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots \mathbf{v}_n\}$ then a linear combination of these vectors is

   $$\mathbf{w} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + \cdots + x_n\mathbf{v}_n$$

   where $x_1, x_2, \ldots, x_n$ are real number constants called weights.

   (c) The Span of a Set of Vectors — The span is the set of all linear combinations of the vectors in the set.

   (d) A Linear Transformation — A linear transformation T is a function such that the following two conditions hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ and scalar c,

      i. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
      ii. $T(c\mathbf{v}) = cT(\mathbf{v})$

   (e) Linear Independence and Dependence — A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots \mathbf{v}_n\}$ is said to be linearly independent if the vector equation,

   $$\mathbf{0} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + \cdots + x_n\mathbf{v}_n$$

   has only the trivial solution for the set of weights, that is, $x_1 = x_2 = \cdots = x_n = 0$. If there is a nontrivial solution then the set is dependent.

2. **True and False:** (*3 Points Each*) Mark each of the following as either true or false. If the statement is false either give a counterexample or correct the statement so that it is true.

   (a) FALSE — The solution set to a linear system involving variables $x_1, \ldots, x_n$ is a list of numbers $(s_1, \ldots, s_n)$ that makes each equation in the system a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$ respectively. — This is a single solution, the solution set is the set of all solutions.

   (b) FALSE — If the solution set to $A\mathbf{x} = \mathbf{0}$ is a plane through the origin then the solution set to $A\mathbf{x} = \mathbf{b}$ is a translation of that plane (assuming $\mathbf{b} \neq \mathbf{0}$). — Only if $A\mathbf{x} = \mathbf{b}$ is consistent.

   (c) TRUE — Elementary row operations on an augmented matrix never change the solution set of the associated linear system.

   (d) FALSE — Elementary row operations can change the pivot positions. They do not.

   (e) FALSE — The row echelon form of a matrix is unique. — The reduced row echelon form of a matrix is unique.

   (f) FALSE — The span of two non-zero vectors in $\mathbb{R}^3$ is a plane through the origin. — Could also be a line.

   (g) TRUE — $A\mathbf{x}$ is a linear combination of the columns of A.

   (h) FALSE — If A is an $m \times n$ matrix whose columns span $\mathbb{R}^m$ then for each vector $\mathbf{b} \in \mathbb{R}^m$ the system $A\mathbf{x} = \mathbf{b}$ has an infinite number of solutions. — It has at least one solution but it may not have infinitely many.

   (i) FALSE — Any subset of a linearly dependent set is dependent. — Subsets of independent sets are independent and supersets of dependent sets are dependent.

   (j) FALSE — The composition of three linear transformations may not result in a linear transformation. — Any composition of linear transformations is a linear transformation.

3. **Calculations:** (*10 Points Each*) Do each of the following.

(a) Write the following system as both a matrix equation and as a vector equation. Then solve the system and put your final answer in parametric vector form. Show each step in the reduction and label the reduction step as we did in class. Finally, write the vector $(-1,-5,-17)$ as a linear combination of the columns of the matrix and describe the solution set geometrically.

$$\begin{aligned} 2x_1 + x_2 + 8x_3 &= -1 \\ 3x_1 + x_2 + 10x_3 &= -5 \\ 13x_1 + 5x_2 + 46x_3 &= -17 \end{aligned}$$

Solution: Matrix and Vector Equations

$$\begin{bmatrix} 2 & 1 & 8 \\ 3 & 1 & 10 \\ 13 & 5 & 46 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ -17 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 3 \\ 13 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} x_2 + \begin{bmatrix} 8 \\ 10 \\ 46 \end{bmatrix} x_3 = \begin{bmatrix} -1 \\ -5 \\ -17 \end{bmatrix}$$

Solving

$$\begin{bmatrix} 2 & 1 & 8 & -1 \\ 3 & 1 & 10 & -5 \\ 13 & 5 & 46 & -17 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 2 & 1 & 8 & -1 \\ 1 & 0 & 2 & -4 \\ 13 & 5 & 46 & -17 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 2 & -4 \\ 2 & 1 & 8 & -1 \\ 13 & 5 & 46 & -17 \end{bmatrix} \xrightarrow{-2R_1+R_2}$$

$$\begin{bmatrix} 1 & 0 & 2 & -4 \\ 0 & 1 & 4 & 7 \\ 13 & 5 & 46 & -17 \end{bmatrix} \xrightarrow{-13R_1+R_3} \begin{bmatrix} 1 & 0 & 2 & -4 \\ 0 & 1 & 4 & 7 \\ 0 & 5 & 20 & 35 \end{bmatrix} \xrightarrow{-5R_2+R_3} \begin{bmatrix} 1 & 0 & 2 & -4 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Parametric Vector Form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4-2x_3 \\ 7-4x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix} x_3$$

Linear Combination

$$\begin{bmatrix} 2 \\ 3 \\ 13 \end{bmatrix}(-4) + \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} 7 + \begin{bmatrix} 8 \\ 10 \\ 46 \end{bmatrix} 0 = \begin{bmatrix} -1 \\ -5 \\ -17 \end{bmatrix}$$

Geometric Interpretation

The set of solutions is a line through the point $(-4,7,0)$ and parallel to the vector $(-2,-4,1)$.

(b) Consider the linear transformation T defined by $T(x_1,x_2,x_3) = (x_1 - 2x_3, x_3 + x_2 - 4x_1)$.

i. What is the domain of T? — $\mathbb{R}^3$

ii. What is the codomain of T? — $\mathbb{R}^2$

iii. If A is the matrix such that $T(\mathbf{x}) = A\mathbf{x}$, what is the size of A? — 2×3

iv. Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

$$\begin{bmatrix} 1 & 0 & -2 \\ -4 & 1 & 1 \end{bmatrix}$$

v. Is T a one-to-one map? Justify your answer. No, since there is not a pivot in each column.

$$\begin{bmatrix} 1 & 0 & -2 \\ -4 & 1 & 1 \end{bmatrix} \xrightarrow{4R_1+R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -7 \end{bmatrix}$$

vi. Is T an onto map? Justify your answer. Yes, since there is a pivot in each row.

(c) For each of the following sets of vectors, tell me if the set is linearly independent or dependent and why. Do as few operations as possible to answer the question.

i. $\{(1,-2),(2,4)\}$ — Independent, two vectors that are not scalar multiples of each other.

ii. $\{(-7,21,-14),(1,-3,2)\}$ — Dependent, these are scalar multiples (-7) of each other.

iii. $\{(2,3,13),(1,1,5),(8,10,46)\}$ — Dependent, these are the vectors from the columns of the coefficient matrix from part a and in that solution we see that there is not a pivot in each column.

iv. $\{(-31,29,33),(41,13,125),(0,47,-17),(13,19,101)\}$ — Dependent, 4 vectors in $\mathbb{R}^3$.

(d) For each of the following sets of vectors, describe geometrically the span of the set of vectors.

i. $\{(1,-2),(2,4)\}$ — $\mathbb{R}^2$

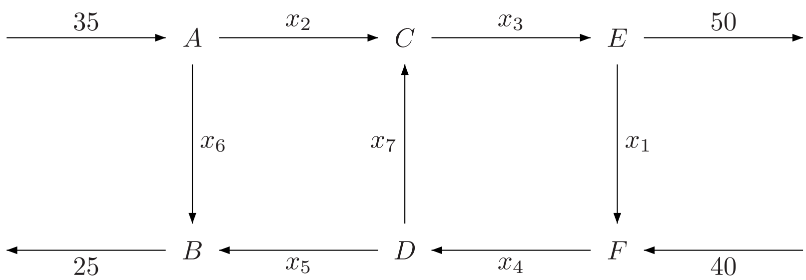
ii. $\{(-7,21,-14),(1,-3,2)\}$ — A line through the origin and the point $(1,-3,2)$.

iii. $\{(2,3,13),(1,1,5),(8,10,46)\}$ — A plane through the origin and the points $(2,3,13)$ and $(1,1,5)$.

iv. $\{(-31,29,33),(41,13,125),(0,47,-17),(13,19,101)\}$ — $\mathbb{R}^3$
Note that the reduced row echelon form of the matrix having this set of vectors as its columns is,

$$\begin{bmatrix} 1 & 0 & 0 & \frac{64311}{136390} \\ 0 & 1 & 0 & \frac{91871}{136390} \\ 0 & 0 & 1 & -\frac{4978}{68195} \end{bmatrix}$$

(e) Given the following network flow, set up the matrix that describes the general flow pattern, do not solve the system.



Solution:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 35 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 25 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 50 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 40 \end{bmatrix}$$

The following is the reduced echelon form of the network flow matrix, what restrictions does this solution place on the traffic flow?

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 & -15 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 35 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 35 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 25 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The parametric solution to the system is,

$$\begin{aligned} x_1 &= -15 - x_6 + x_7 \\ x_2 &= 35 - x_6 \\ x_3 &= 35 - x_6 + x_7 \\ x_4 &= 25 - x_6 + x_7 \\ x_5 &= 25 - x_6 \end{aligned}$$

So, we know that $x_7 - x_6 \geq 15$ and $x_6 \leq 25$.

(f) In a certain region, about 7% of a city's population moves to the surrounding suburbs each year, and about 5% of the suburban population moves into the city. In 2010, there were 800,000 residents in the city and 500,000 in the suburbs. Set up the migration matrix for this situation and estimate the populations of the city and suburbs in 2012.

Solution: The migration matrix is

$$\begin{bmatrix} 0.93 & 0.05 \\ 0.07 & 0.95 \end{bmatrix}$$

So in 2011 we have

$$\begin{bmatrix} 0.93 & 0.05 \\ 0.07 & 0.95 \end{bmatrix} \begin{bmatrix} 800000 \\ 500000 \end{bmatrix} = \begin{bmatrix} 769000.0 \\ 531000.0 \end{bmatrix}$$

and in 2012 we have

$$\begin{bmatrix} 0.93 & 0.05 \\ 0.07 & 0.95 \end{bmatrix} \begin{bmatrix} 769000.0 \\ 531000.0 \end{bmatrix} = \begin{bmatrix} 741720.0 \\ 558280.0 \end{bmatrix}$$