Spring 2012 **Exam #2 Key** Math 306

Exam #2 Key

1. **Definitions:** (*4 Points Each*) Give a definition for each of the following.

   (a) An Elementary Matrix — An Elementary Matrix is produced by taking the $n \times n$ identity matrix and doing one row operation on it.

   (b) The Null Space of a matrix — The Null Space of a matrix A is the set of solutions to the equation $A\mathbf{x} = \mathbf{b}$.

   (c) A Subspace of $\mathbb{R}^n$ — A Subspace of $\mathbb{R}^n$ is a subset H such that $\mathbf{0} \in H$, H is closed under vector addition and scalar multiplication.

   (d) The Rank of a matrix — The Rank of a matrix A is the dimension of the column space of A.

   (e) The (i,j)-Cofactor, C_{ij}, of a matrix A — The (i,j)-Cofactor, C_{ij}, of a matrix A is $C_{ij} = (-1)^{i+j}\det(A_{ij})$, where A_{ij} is the (i,j)-Minor of A, which is created by deleting the i^{th} row and j^{th} column of A.

2. **True and False:** (*3 Points Each*) Mark each of the following as either true or false. If the statement is false either give a counterexample or correct the statement so that it is true.

   (a) False — $\det(A+B) = \det(A) + \det(B)$, where both A and B are $n \times n$ matrices. — $\det(AB) = \det(A)\det(B)$

   (b) False — The determinant of a triangular matrix is the sum of the entries on the main diagonal. — It is the *product* of the entries on the main diagonal.

   (c) True — If A is an invertible matrix then $\det(A^{-1}) = (\det(A))^{-1}$.

   (d) False — If A is a square matrix such that $A^T A = I$ then $\det(A) = 1$. — $\det(A) = \pm 1$

   (e) False — $(AB)^T = A^T B^T$ for matrices A and B such that AB is defined. — $(AB)^T = B^T A^T$

   (f) True — If $AB = 0$ for non-zero $n \times n$ matrices A and B, then neither A nor B can be invertible.

   (g) True — If A is an $n \times n$ matrix such that the columns of A span $\mathbb{R}^n$ then the columns of A are linearly independent and form a basis for $\mathbb{R}^n$.

   (h) True — If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} \subset \mathbb{R}^n$ then $\mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\})$ is a subspace of $\mathbb{R}^n$.

   (i) False — If T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $T\mathbf{x} = A\mathbf{x}$ and the dimension of the Null Space of A is 0 then T is an onto map. — T is an one-to-one map.

   (j) True — If B is the reduced echelon form of the matrix A then $B = MA$ for some invertible matrix M.

3. **Calculations:** (*10 Points Each*) Do each of the following.

   (a) Given that

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ -2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 3 & 5 \\ 2 & -2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

   find the following if they exist. If the operation is not defined state why?

   i. $2A - 3B$ — Undefined, matrix sizes are not the same.

   ii. $AB = \begin{bmatrix} 5 & -3 & 8 \\ -2 & 10 & 21 \\ 12 & -16 & -5 \end{bmatrix}$

   iii. BC — Undefined, B is 2×3 and C is 2×3 so the columns of the first do not match the rows of the second.

   iv. $BAC = \begin{bmatrix} 1 & 28 & -24 \\ -8 & -15 & -17 \end{bmatrix}$

   v. $2A - 3C^T = \begin{bmatrix} -1 & 6 \\ -1 & -1 \\ -7 & 13 \end{bmatrix}$

(b) Find the determinant of the following matrix, A. You may use a short-cut method if it is applicable. Is the matrix A invertible? If so find its inverse and if not state why. Show all of the steps in the derivations and keep your answers in exact form.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 3 \\ -2 & 1 & -2 \end{bmatrix}$$

Solution: $|A| = 15$ so A is invertible.

$$\left[\begin{array}{cccccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 3 & 1 & 3 & 0 & 1 & 0 \\ -2 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cccccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -5 & -9 & -3 & 1 & 0 \\ -2 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{2R_1+R_3} \left[\begin{array}{cccccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -5 & -9 & -3 & 1 & 0 \\ 0 & 5 & 6 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{cccccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{9}{5} & \frac{3}{5} & -\frac{1}{5} & 0 \\ 0 & 5 & 6 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_1} \left[\begin{array}{cccccc} 1 & 0 & \frac{2}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & \frac{9}{5} & \frac{3}{5} & -\frac{1}{5} & 0 \\ 0 & 5 & 6 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-5R_2+R_3} \left[\begin{array}{cccccc} 1 & 0 & \frac{2}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & \frac{9}{5} & \frac{3}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & -3 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_3} \left[\begin{array}{cccccc} 1 & 0 & \frac{2}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & \frac{9}{5} & \frac{3}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

$$\xrightarrow{-\frac{9}{5}R_3+R_2} \left[\begin{array}{cccccc} 1 & 0 & \frac{2}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & 0 & 0 & \frac{2}{5} & \frac{3}{5} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

$$\xrightarrow{-\frac{2}{5}R_3+R_1} \left[\begin{array}{cccccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{8}{15} & \frac{2}{15} \\ 0 & 1 & 0 & 0 & \frac{2}{5} & \frac{3}{5} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{8}{15} & \frac{2}{15} \\ 0 & \frac{2}{5} & \frac{3}{5} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

(c) Find the determinant of the following matrix using cofactor expansion. You may use a short-cut method if and when it is applicable. Show all of the steps in the derivations and keep your answers in exact form.

$$A = \begin{bmatrix} -4 & 0 & -9 & -2 \\ -5 & -3 & -4 & 0 \\ 8 & 1 & 10 & 5 \\ -4 & -10 & -1 & 3 \end{bmatrix}$$

Solution:

$$\begin{aligned} |A| &= -4(-1)^{1+1}\begin{vmatrix} -3 & -4 & 0 \\ 1 & 10 & 5 \\ -10 & -1 & 3 \end{vmatrix} + (-9)(-1)^{1+3}\begin{vmatrix} -5 & -3 & 0 \\ 8 & 1 & 5 \\ -4 & -10 & 3 \end{vmatrix} + (-2)(-1)^{1+4}\begin{vmatrix} -5 & -3 & -4 \\ 8 & 1 & 10 \\ -4 & -10 & -1 \end{vmatrix} \\ &= -4 \cdot 107 - 9 \cdot (-133) + 2 \cdot (-95) \\ &= 579 \end{aligned}$$

(d) Find the determinant of the following matrix using the reduction method. You may not use a short-cut method nor a combination of reduction and cofactor expansion. The reduction method should be carried out to at least echelon form. Show all of the steps in the derivations and keep your answers in exact form.

$$A = \begin{bmatrix} -4 & 0 & -9 & -2 \\ -5 & -3 & -4 & 0 \\ 8 & 1 & 10 & 5 \\ -4 & -10 & -1 & 3 \end{bmatrix}$$

Solution:

$$\begin{aligned} \begin{vmatrix} -4 & 0 & -9 & -2 \\ -5 & -3 & -4 & 0 \\ 8 & 1 & 10 & 5 \\ -4 & -10 & -1 & 3 \end{vmatrix} &= \begin{vmatrix} 1 & 3 & -5 & -2 \\ -5 & -3 & -4 & 0 \\ 8 & 1 & 10 & 5 \\ -4 & -10 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -5 & -2 \\ 0 & 12 & -29 & -10 \\ 0 & -23 & 50 & 21 \\ 0 & 2 & -21 & -5 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 3 & -5 & -2 \\ 0 & 12 & -29 & -10 \\ 0 & 1 & -8 & 1 \\ 0 & 2 & -21 & -5 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & -5 & -2 \\ 0 & 1 & -8 & 1 \\ 0 & 12 & -29 & -10 \\ 0 & 2 & -21 & -5 \end{vmatrix} \\ &= -\begin{vmatrix} 1 & 0 & 19 & -5 \\ 0 & 1 & -8 & 1 \\ 0 & 0 & 67 & -22 \\ 0 & 0 & -5 & -7 \end{vmatrix} = -67\begin{vmatrix} 1 & 0 & 19 & -5 \\ 0 & 1 & -8 & 1 \\ 0 & 0 & 1 & -\frac{22}{67} \\ 0 & 0 & -5 & -7 \end{vmatrix} \\ &= -67\begin{vmatrix} 1 & 0 & 19 & -5 \\ 0 & 1 & -8 & 1 \\ 0 & 0 & 1 & -\frac{22}{67} \\ 0 & 0 & 0 & -\frac{579}{67} \end{vmatrix} = -67 \cdot \left(-\frac{579}{67}\right) = 579 \end{aligned}$$

(e) For the following matrix A find a basis for Nul(A) and Col(A). Show all of the steps in the derivations and keep your answers in exact form.

$$A = \begin{bmatrix} 2 & 4 & -2 & 2 & 3 \\ -1 & 1 & -2 & 1 & 1 \\ 3 & 5 & -2 & -1 & 3 \end{bmatrix}$$

Solution:

$$\begin{aligned}
\begin{bmatrix} 2 & 4 & -2 & 2 & 3 \\ -1 & 1 & -2 & 1 & 1 \\ 3 & 5 & -2 & -1 & 3 \end{bmatrix}
&\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 1 & -2 & 1 & 1 \\ 2 & 4 & -2 & 2 & 3 \\ 3 & 5 & -2 & -1 & 3 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 4 & -2 & 2 & 3 \\ 3 & 5 & -2 & -1 & 3 \end{bmatrix} \\
&\xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 6 & -6 & 4 & 5 \\ 3 & 5 & -2 & -1 & 3 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 6 & -6 & 4 & 5 \\ 0 & 8 & -8 & 2 & 6 \end{bmatrix} \\
&\xrightarrow{\frac{1}{6}R_2} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -1 & \frac{2}{3} & \frac{5}{6} \\ 0 & 8 & -8 & 2 & 6 \end{bmatrix} \xrightarrow{-8R_2+R_3} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -1 & \frac{2}{3} & \frac{5}{6} \\ 0 & 0 & 0 & -\frac{10}{3} & -\frac{2}{3} \end{bmatrix} \\
&\xrightarrow{-\frac{3}{10}R_3} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -1 & \frac{2}{3} & \frac{5}{6} \\ 0 & 0 & 0 & 1 & \frac{1}{5} \end{bmatrix} \xrightarrow{-\frac{2}{3}R_3+R_2} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -1 & 0 & \frac{7}{10} \\ 0 & 0 & 0 & 1 & \frac{1}{5} \end{bmatrix} \\
&\xrightarrow{R_3+R_1} \begin{bmatrix} 1 & -1 & 2 & 0 & -\frac{4}{5} \\ 0 & 1 & -1 & 0 & \frac{7}{10} \\ 0 & 0 & 0 & 1 & \frac{1}{5} \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 1 & 0 & 1 & 0 & -\frac{1}{10} \\ 0 & 1 & -1 & 0 & \frac{7}{10} \\ 0 & 0 & 0 & 1 & \frac{1}{5} \end{bmatrix}
\end{aligned}$$

So a basis to the column space is

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Solving the system gives

$$\mathbf{x} = \begin{bmatrix} -x_3 + \frac{1}{10}x_5 \\ x_3 - \frac{7}{10}x_5 \\ x_3 \\ -\frac{1}{5}x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} \frac{1}{10} \\ -\frac{7}{10} \\ 0 \\ -\frac{1}{5} \\ 1 \end{bmatrix} x_5$$

and a basis to the null space is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{10} \\ -\frac{7}{10} \\ 0 \\ -\frac{1}{5} \\ 1 \end{bmatrix} \right\}$$

(f) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ where

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \qquad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{b}_3 = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$$

be a basis for $\mathbb{R}^3$ and let

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

find $[\mathbf{x}]_{\mathcal{B}}$. Show all of the steps in the derivations and keep your answers in exact form.

Solution:

$$\begin{aligned}
\begin{bmatrix} 1 & 2 & 4 & 1 \\ 3 & 1 & 3 & 2 \\ -2 & 1 & -1 & 3 \end{bmatrix} & \xrightarrow{-3R_1+R_2} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -5 & -9 & -1 \\ -2 & 1 & -1 & 3 \end{bmatrix} \xrightarrow{2R_1+R_3} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -5 & -9 & -1 \\ 0 & 5 & 7 & 5 \end{bmatrix} \\
& \xrightarrow{R_2+R_3} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -5 & -9 & -1 \\ 0 & 0 & -2 & 4 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -5 & -9 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \\
& \xrightarrow{9R_3+R_2} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -5 & 0 & -19 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{-4R_3+R_1} \begin{bmatrix} 1 & 2 & 0 & 9 \\ 0 & -5 & 0 & -19 \\ 0 & 0 & 1 & -2 \end{bmatrix} \\
& \xrightarrow{-\frac{1}{5}R_2} \begin{bmatrix} 1 & 2 & 0 & 9 \\ 0 & 1 & 0 & \frac{19}{5} \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & \frac{7}{5} \\ 0 & 1 & 0 & \frac{19}{5} \\ 0 & 0 & 1 & -2 \end{bmatrix}
\end{aligned}$$

So

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{7}{5} \\ \frac{19}{5} \\ -2 \end{bmatrix}$$