Spring 2012

Exam #3

Math 306

Name: ____

Write all of your responses on these exam pages. If you need more space please use the backs. Make sure that you show all of your work.

1. **Definitions:** (*3 Points Each*) Give a definition for each of the following.

   (a) An Eigenvalue and Eigenvector of a Matrix

   (b) The Kernel of a Linear Transformation

   (c) A Subspace of a vector space V

   (d) An Isomorphism of two vector spaces V and W

   (e) The Dimension of a Vector Space

2. **True and False:** (*2 Points Each*) Mark each of the following as either true or false. If the statement is false either give a counterexample or correct the statement so that it is true. The insertion of the word *not* or changing an $=$ to $\neq$ is insufficient for correcting a statement.

   (a) ________ $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = P_{\mathcal{B}} P_{\mathcal{C}}^{-1}$

   (b) ________ The rank of a matrix is the dimension of the row space of the matrix.

   (c) ________ If H is a subspace of a vector space V then $\dim(H) < \dim(V)$.

   (d) ________ The vector space $\mathbb{P}_3$ is isomorphic to a subspace of $\mathbb{R}^6$.

   (e) ________ Given a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ of a vector space V the coordinate map $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is an isomorphism from V to $\mathbb{R}^n$.

   (f) ________ If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors then they correspond to distinct eigenvalues.

   (g) ________ A nilpotent matrix is a square matrix such that $A^n = 0$ for some power n. The only eigenvalue of a nilpotent matrix A is 0.

   (h) ________ If $A = PBP^{-1}$ for some invertible matrix P then the characteristic polynomials of A and B could be different but A and B will have the same eigenvalues.

   (i) ________ If a 4×4 matrix A has eigenvalues 2, 3, $-\frac{2}{3}$ and -21 then A is diagonalizable.

   (j) ________ The dimension of an eigenspace for an eigenvalue λ is always less than or equal to the algebraic multiplicity of the eigenvalue λ.

3. **Proofs:** (*10 Points Each*) Prove each of the following.

   (a) Let T be a one-to-one linear transformation from V to W and let $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a linearly independent set of vectors in V. Show that the set $\{T(\mathbf{b}_1), T(\mathbf{b}_2), \ldots, T(\mathbf{b}_n)\}$ is a linearly independent set of vectors in W. Hint: you may want to prove the contrapositive of this statement.

   (b) Show that if A is an invertible matrix then it cannot have 0 as an eigenvalue. Then show that if A is an invertible matrix and λ is an eigenvalue of A then λ^{-1} is an eigenvalue of A^{-1}.

4. **Calculations:** Do each of the following. For full credit you must show all of the steps in each derivation.

   (a) (*20 Points*) Consider the following matrix, A,

$$A = \begin{bmatrix} -9 & 15 & 3 \\ -12 & 18 & 3 \\ 24 & -30 & -3 \end{bmatrix}$$

      i. Find the characteristic polynomial for A.

      ii. Find the eigenvalues and their multiplicities for A. (Hint: the eigenvalues are "nice" numbers)

iii. Find bases for each eigenspace of A.

iv. Is A diagonalizable? If so find D and P such that $A = PDP^{-1}$ and if not explain why.

(b) (*20 Points*) Let $T : \mathbb{P}_2 \to \mathbb{R}^3$ be defined as

$$T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}'(0) \\ \mathbf{p}(1) \end{bmatrix}$$

where $\mathbf{p}'$ is the derivative of $\mathbf{p}$.

i. Show that T is a linear transformation.

ii. Find $\ker(T)$.

iii. Find a basis to the range of T.

iv. Is T one-to-one? Verify your answer.

v. Is T onto? Verify your answer.

vi. Is T an isomorphism? Verify your answer.

(c) (*10 Points*) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ with

$$\mathbf{b}_1 = \begin{bmatrix} 6 \\ -12 \end{bmatrix} \qquad \mathbf{b}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \qquad \mathbf{c}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \qquad \mathbf{c}_2 = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

Find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ and $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$.