Spring 2012

Exam #3 Key

Math 306

1. **Definitions:** (*3 Points Each*) Give a definition for each of the following.

   (a) An Eigenvalue and Eigenvector of a Matrix — For a square matrix A if there is a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ then $\mathbf{x}$ is an eigenvector of A and λ is an eigenvalue of A.

   (b) The Kernel of a Linear Transformation — Let T be a linear transformation then $\ker(T) = \{\mathbf{x} \mid T(\mathbf{x}) = \mathbf{0}\}$.

   (c) A Subspace of a vector space V — A subset H of a vector space V is a subspace of V if

      i. $\mathbf{0} \in H$

      ii. For all vectors $\mathbf{x}$ and $\mathbf{y}$ in H, $\mathbf{x} + \mathbf{y} \in H$.

      iii. For all vectors $\mathbf{x} \in H$ and scalars $c \in \mathbb{R}$, $c\mathbf{x} \in H$.

   (d) An Isomorphism of two vector spaces V and W — An Isomorphism is a one-to-one and onto linear transformation.

   (e) The Dimension of a Vector Space — The dimension of a vector space is the number of vectors in any basis for the vector space if the basis is finite. If any basis is infinite we say that the vector space is infinite dimensional.

2. **True and False:** (*2 Points Each*) Mark each of the following as either true or false. If the statement is false either give a counterexample or correct the statement so that it is true. The insertion of the word *not* or changing an $=$ to $\neq$ is insufficient for correcting a statement.

   (a) **FALSE:** $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = P_{\mathcal{B}} P_{\mathcal{C}}^{-1}$

   Solution: $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = P_{\mathcal{C}} P_{\mathcal{B}}^{-1}$

   (b) **TRUE:** The rank of a matrix is the dimension of the row space of the matrix.

   (c) **FALSE:** If H is a subspace of a vector space V then $\dim(H) < \dim(V)$.
   Solution: $\dim(H) \leq \dim(V)$

   (d) **TRUE:** The vector space $\mathbb{P}_3$ is isomorphic to a subspace of $\mathbb{R}^6$.

   (e) **TRUE:** Given a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ of a vector space V the coordinate map $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is an isomorphism from V to $\mathbb{R}^n$.

   (f) **FALSE:** If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors then they correspond to distinct eigenvalues.
   Solution: If $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors that correspond to distinct eigenvalues then they are linearly independent.

   (g) **TRUE:** A nilpotent matrix is a square matrix such that $A^n = 0$ for some power n. The only eigenvalue of a nilpotent matrix A is 0.

   (h) **FALSE:** If $A = PBP^{-1}$ for some invertible matrix P then the characteristic polynomials of A and B could be different but A and B will have the same eigenvalues.
   Solution: The characteristic polynomials of A and B will also be identical.

   (i) **TRUE:** If a 4×4 matrix A has eigenvalues 2, 3, $-\frac{2}{3}$ and -21 then A is diagonalizable.

   (j) **TRUE:** The dimension of an eigenspace for an eigenvalue λ is always less than or equal to the algebraic multiplicity of the eigenvalue λ.

3. **Proofs:** (*10 Points Each*) Prove each of the following.

   (a) Let T be a one-to-one linear transformation from V to W and let $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a linearly independent set of vectors in V. Show that the set $\{T(\mathbf{b}_1), T(\mathbf{b}_2), \ldots, T(\mathbf{b}_n)\}$ is a linearly independent set of vectors in W. Hint: you may want to prove the contrapositive of this statement.

   Solution: Proving the contrapositive of this statement, assume that the set $\{T(\mathbf{b}_1), T(\mathbf{b}_2), \ldots, T(\mathbf{b}_n)\}$ is linearly dependent. Then there are weights c_1, c_2, $\ldots$, c_n, not all zero, such that

$$c_1T(\mathbf{b}_1) + c_2T(\mathbf{b}_2) + \cdots + c_nT(\mathbf{b}_n) = \mathbf{0}$$

   Using the fact that T is a linear transformation, the above equation becomes

$$T(c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n) = \mathbf{0}$$

   Since T is one-to-one we also know that $\ker(T) = \mathbf{0}$, so

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n = \mathbf{0}$$

   and since we know that not all of the weights are 0, we have a dependency relation for $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and hence this set is dependent.

   (b) Show that if A is an invertible matrix then it cannot have 0 as an eigenvalue. Then show that if A is an invertible matrix and λ is an eigenvalue of A then λ^{-1} is an eigenvalue of A^{-1}.

   Solution: If A is invertible and has an eigenvalue of 0 then there is a non-zero vector $\mathbf{x}$ with $A\mathbf{x} = \mathbf{0}$ but by the Invertible Matrix Theorem the null space of A must be trivial, a contradiction. Now assume that λ is an eigenvalue of A, then there is a non-zero vector $\mathbf{x}$ with $A\mathbf{x} = \lambda\mathbf{x}$. Multiplying both sides by A^{-1} gives $A^{-1}A\mathbf{x} = A^{-1}\lambda\mathbf{x}$ and hence $\mathbf{x} = \lambda A^{-1}\mathbf{x}$. Since $\lambda \neq 0$ we have $A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$, showing that λ^{-1} is an eigenvalue for A^{-1}.

4. **Calculations:** Do each of the following. For full credit you must show all of the steps in each derivation.

   (a) (*20 Points*) Consider the following matrix, A,

$$A = \begin{bmatrix} -9 & 15 & 3 \\ -12 & 18 & 3 \\ 24 & -30 & -3 \end{bmatrix}$$

   i. Find the characteristic polynomial for A.
   Solution: $-x^3 + 6x^2 - 9x$

   ii. Find the eigenvalues and their multiplicities for A. (Hint: the eigenvalues are "nice" numbers)
   Solution: The eigenvalues are 0 and 3, 0 has multiplicity 1 and 3 has multiplicity 2.

   iii. Find bases for each eigenspace of A.
   Solution: For $\lambda = 0$ a basis is $\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$ and For $\lambda = 3$ a basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} \right\}$

   iv. Is A diagonalizable? If so find D and P such that $A = PDP^{-1}$ and if not explain why.
   Solution: Yes it is, since A has three linearly independent eigenvectors.

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 4 & -5 & -2 \end{bmatrix}$$

(b) (*20 Points*) Let $T : \mathbb{P}_2 \to \mathbb{R}^3$ be defined as

$$T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}'(0) \\ \mathbf{p}(1) \end{bmatrix}$$

where $\mathbf{p}'$ is the derivative of $\mathbf{p}$.

i. Show that T is a linear transformation.
Solution: Let $\mathbf{p}, \mathbf{q} \in \mathbb{P}_2$ then

$$T(\mathbf{p}+\mathbf{q}) = \begin{bmatrix} (\mathbf{p}+\mathbf{q})(0) \\ (\mathbf{p}+\mathbf{q})'(0) \\ (\mathbf{p}+\mathbf{q})(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0)+\mathbf{q}(0) \\ \mathbf{p}'(0)+\mathbf{q}'(0) \\ \mathbf{p}(1)+\mathbf{q}(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}'(0) \\ \mathbf{p}(1) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(0) \\ \mathbf{q}'(0) \\ \mathbf{q}(1) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q})$$

and

$$T(c\mathbf{p}) = \begin{bmatrix} c\mathbf{p}(0) \\ c\mathbf{p}'(0) \\ c\mathbf{p}(1) \end{bmatrix} = c \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}'(0) \\ \mathbf{p}(1) \end{bmatrix} = cT(\mathbf{p})$$

ii. Find $\ker(T)$.
Solution: Let $\mathbf{p} \in \ker(T)$ and write $\mathbf{p}(t) = at^2 + bt + c$. So

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}'(0) \\ \mathbf{p}(1) \end{bmatrix} = \begin{bmatrix} c \\ b \\ a+b+c \end{bmatrix}$$

The first two components gives $c = b = 0$ and substitution of these into the third component gives $a = 0$ as well. So $\mathbf{p}(t) = 0$ and hence $\ker(T) = \{\mathbf{0}\}$.

iii. Find a basis to the range of T.
Solution: Again let $\mathbf{p}(t) = at^2 + bt + c$ then

$$T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}'(0) \\ \mathbf{p}(1) \end{bmatrix} = \begin{bmatrix} c \\ b \\ a+b+c \end{bmatrix} = c \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + a \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so a basis for the range of T is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

iv. Is T one-to-one? Verify your answer.
Solution: Yes, the kernel of T is trivial.

v. Is T onto? Verify your answer.
Solution: Yes, a basis for the range consists of 3 vectors which thus span $\mathbb{R}^3$.

vi. Is T an isomorphism? Verify your answer.
Solution: Yes, since T is a one-to-one and onto linear transformation it is by definition an isomorphism.

(c) (*10 Points*) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ with

$$\mathbf{b}_1 = \begin{bmatrix} 6 \\ -12 \end{bmatrix} \qquad \mathbf{b}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \qquad \mathbf{c}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \qquad \mathbf{c}_2 = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

Find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ and $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$.

Solution: To find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$

$$\begin{bmatrix} 4 & 3 & 6 & 4 \\ 2 & 9 & -12 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

so

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$$

To find $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$

$$\begin{bmatrix} 6 & 4 & 4 & 3 \\ -12 & 2 & 2 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 & \frac{3}{2} \end{bmatrix}$$

so

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{3}{2} \end{bmatrix}$$