Spring 2012

Final Exam

Math 306

Name: ______________________________

Write all of your responses on these exam pages. If you need more space please use the backs. Make sure that you show all of your work.

1. **Definitions:** (*4 Points Each*) Give a definition for each of the following.

   (a) A Linear Transformation

   (b) Linear Independence and Dependence

   (c) A Basis of a vector space V

   (d) An Eigenvalue and Eigenvector of a matrix

(e) The Kernel of a linear transformation

(f) A Subspace of a vector space V

(g) The Dimension of a vector space V

(h) The Orthogonal Complement of a subspace W of $\mathbb{R}^n$

2. **True and False:** (*2 Points Each*) Mark each of the following as either true or false. If the statement is false either give a counterexample or correct the statement so that it is true. The insertion of the word *not* or changing an $=$ to $\neq$ is insufficient for correcting a statement.

   (a) ________ The solution set to a linear system involving variables $x_1, \ldots, x_n$ is the set of all lists of numbers $(s_1, \ldots, s_n)$ that makes each equation in the system a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$ respectively.

   (b) ________ If every column of the coefficient matrix of an augmented matrix contains a pivot then the system is consistent.

   (c) ________ The system $[\,\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ \ \mathbf{b}\,]$ is consistent if and only if $\mathbf{b} \in \mathrm{Span}(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\})$.

   (d) ________ If $A\mathbf{x} = \mathbf{b}$ is consistent then any solution to the system can be written as $\mathbf{w} = \mathbf{p} + \mathbf{v}$, where $\mathbf{v}$ is a solution to the corresponding homogeneous system and $\mathbf{p}$ is a single fixed vector.

   (e) ________ If four distinct vectors in $\mathbb{R}^n$ lie in the same plane then they must be linearly dependent and a subset of two of them will form a basis to the span of the four vectors.

   (f) ________ Any linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be written as an $n \times m$ matrix.

   (g) ________ A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if given any vector $\mathbf{y} \in \mathbb{R}^m$ there is one vector $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{y}$.

   (h) ________ $(AB^{-1}C)^T = C^T(B^T)^{-1}A^T$

   (i) ________ If A is a matrix such that $A\mathbf{x} = \mathbf{e}_1$ has an infinite number of solutions, then there exists a number j with $2 \leq j \leq n$ where $A\mathbf{x} = \mathbf{e}_j$ is inconsistent.

   (j) ________ If A^T is not invertible then A is not invertible.

(k) ________ If A is an $n \times m$ matrix then the set of all linear combinations of the rows of A form a subspace of $\mathbb{R}^n$.

(l) ________ The rank of a matrix A is the dimension of the row space of A.

(m) ________ If a matrix B is obtained from a square matrix A by doing a row swap, then a column swap and finally multiplying one row by a non-zero constant and adding the result to another row, then $|B| = |A|$.

(n) ________ The Nul(A) is the kernel of the transformation $T(\mathbf{x}) = A\mathbf{x}$.

(o) ________ If $AP = PD$ where D is diagonal and P is invertible then the columns of P are eigenvectors of A and the diagonal entries of D are the roots of the characteristic polynomial of A.

(p) ________ If $P_{\mathcal{B}}$ is the change of coordinate matrix then $P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$ for all $\mathbf{x} \in V$.

(q) ________ A vector space is infinite dimensional if there does not exist a finite subset of vectors that span the space.

(r) ________ If U is a matrix with orthonormal columns then the transformation $T(\mathbf{x}) = U\mathbf{x}$ preserves both length and orthogonality.

(s) ________ A square matrix A is not invertible if and only if 0 is an eigenvalue of A.

(t) ________ If $\mathbf{x}$ is orthogonal to each of the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, then $\mathbf{x}$ must be orthogonal to every linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

3. **Proofs:** (*10 Points Each*) Prove each of the following.

   (a) Prove that if A is an invertible matrix then $|A| \neq 0$.

   (b) Show that λ is an eigenvalue of a matrix A if and only if λ is an eigenvalue of A^T.

(c) Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_t\}$ and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q\}$ be an orthogonal basis to $W^\perp$.

  i. Explain why $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_t, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q\}$ is an orthogonal set.

  ii. Explain why $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_t, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q\}$ spans $\mathbb{R}^n$.

  iii. Show that $\dim(W) + \dim(W^\perp) = n$

4. **Calculations:** (*15 Points Each*) Do each of the following. For full credit you must show all of the steps in each derivation.

   (a) Solve the following system of equations and put your final answer in parametric vector form.

$$\begin{aligned}
3x_1 - x_2 + 11x_3 + 4x_4 &= 18 \\
-2x_1 + x_2 - 8x_3 - 3x_4 &= -11 \\
-5x_1 + 3x_2 - 21x_3 - 8x_4 &= -26 \\
3x_1 - 2x_2 + 13x_3 + 5x_4 &= 15
\end{aligned}$$

(b) Find the inverse of the following matrix

$$\begin{bmatrix} 3 & -1 & 11 \\ -2 & 1 & -8 \\ -5 & 3 & -20 \end{bmatrix}$$

(c) Use cofactor expansion, reduction or a combination of both to find the determinant of the following matrix. Once you are to a 3×3 matrix you may use the short-cut method.

$$\begin{bmatrix} 9 & -2 & -7 & -8 & 9 \\ 0 & 0 & -7 & 1 & 0 \\ 0 & 2 & -1 & 5 & 1 \\ 1 & -5 & -10 & 4 & 6 \\ 0 & -8 & 8 & 2 & -3 \end{bmatrix}$$

(d) Find bases for Nul(A) and Col(A) for the given matrix A.

$$A = \begin{bmatrix} 3 & -1 & 11 & 4 & 18 \\ -2 & 1 & -8 & -3 & -11 \\ -5 & 3 & -21 & -8 & -26 \\ 3 & -2 & 13 & 5 & 15 \end{bmatrix}$$

(e) Let $T : \mathbb{P}_3 \to \mathbb{R}^3$ be defined as

$$T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}'(0) \\ \mathbf{p}''(0) \end{bmatrix}$$

where $\mathbf{p}'$ and $\mathbf{p}''$ are the first ans second derivatives of $\mathbf{p}$.

i. Show that T is a linear transformation.

ii. Find a basis to the $\ker(T)$.

iii. Find a basis to the range of T.

iv. Is T one-to-one? Verify your answer.

v. Is T onto? Verify your answer.

vi. Is T an isomorphism? Verify your answer.

(f) For the following matrix A,

$$A = \begin{bmatrix} 24 & 88 & -22 \\ -16 & -62 & 16 \\ -40 & -160 & 42 \end{bmatrix}$$

i. Find the characteristic polynomial for A.

ii. Find the eigenvalues and their multiplicities for A. (Hint: the eigenvalues are “nice” numbers)

iii. Find bases for each eigenspace of A.

iv. Is A diagonalizable? If so find D and P such that $A = PDP^{-1}$ and if not explain why.

(g) Let

$$\mathbf{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

and let $W = \text{Span}(\{\mathbf{u}, \mathbf{v}\})$. Write $\mathbf{y} = \mathbf{x} + \mathbf{z}$ where $\mathbf{x} \in W$ and $\mathbf{z} \in W^{\perp}$.