1. (*30 points*): Mark each of the following as either True or False by circling the correct answer.

   (a) **True False**: Elementary row operations on an augmented matrix never change the solution set of the associated linear system. **Solution:** True

   (b) **True False**: Two matrices are row equivalent if they have the same number of rows. **Solution:** False

   (c) **True False**: The pivot positions in a matrix depend on whether row interchanges are used in the row reduction process. **Solution:** False

   (d) **True False**: Suppose a system of linear equations has a 3×5 augmented matrix whose fifth column is a pivot column. The system consistent? **Solution:** False

   (e) **True False**: When $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ contains the line through $\mathbf{u}$ and the origin. **Solution:** True

   (f) **True False**: The equation $A\mathbf{x} = \mathbf{b}$ is consistent if the augmented matrix $[A\ \mathbf{b}]$ has a pivot position in every row. **Solution:** False

   (g) **True False**: The equation $A\mathbf{x} = \mathbf{b}$ is homogeneous if the zero vector is a solution. **Solution:** True

   (h) **True False**: The solution set of a consistent system $A\mathbf{x} = \mathbf{b}$ is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$. **Solution:** True

   (i) **True False**: The columns of the matrix A are linearly independent if the equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution. **Solution:** False

   (j) **True False**: The columns of any 4×5 matrix are linearly dependent. **Solution:** True

2. (*40 points*): For the following system of linear equations,

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 4x_1 + 5x_2 + 6x_3 &= 7 \\ 6x_1 + 7x_2 + 8x_3 &= 9 \end{aligned}$$

   (a) Write the system in matrix equation form $A\mathbf{x} = \mathbf{b}$, explicitly denote A, $\mathbf{x}$, and $\mathbf{b}$.
   Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 9 \end{bmatrix}$$

   (b) Construct the associated augmented matrix for the system.
   Solution:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$$

(c) Reduce the augmented matrix to reduced row echelon form.

Solution:

$$\begin{bmatrix}1&2&3&4\\4&5&6&7\\6&7&8&9\end{bmatrix} \sim \begin{bmatrix}1&2&3&4\\0&-3&-6&-9\\6&7&8&9\end{bmatrix} \sim \begin{bmatrix}1&2&3&4\\0&-3&-6&-9\\0&-5&-10&-15\end{bmatrix} \sim \begin{bmatrix}1&2&3&4\\0&1&2&3\\0&-5&-10&-15\end{bmatrix}$$

$$\sim \begin{bmatrix}1&2&3&4\\0&1&2&3\\0&0&0&0\end{bmatrix} \sim \begin{bmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&0\end{bmatrix}$$

(d) Is the system consistent or inconsistent?

Solution: Consistent

(e) If the system is consistent,

i. Write the solution in parametric form.
Solution:

$$\begin{aligned}x_1 &= -2 + x_3\\ x_2 &= 3 - 2x_3\\ x_3 &= x_3\end{aligned}$$

ii. Write the solution in parametric vector form.
Solution:

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2\\3\\0\end{bmatrix} + x_3\begin{bmatrix}1\\-2\\1\end{bmatrix}$$

iii. What are the basic variables and what are the free variables?
Solution: x_1 and x_2 are basic and x_3 is free.

iv. Describe the set of solutions geometrically.
Solution: This is the straight line in $\mathbb{R}^3$ through the points $(-2, 3, 0)$ and $(-1, 1, 1)$.

3. (*10 points*): Compute

$$\begin{bmatrix}6&5\\-4&-3\\7&6\end{bmatrix}\begin{bmatrix}1\\-3\end{bmatrix}$$

Solution:

$$\begin{bmatrix}6&5\\-4&-3\\7&6\end{bmatrix}\begin{bmatrix}1\\-3\end{bmatrix} = \begin{bmatrix}-9\\5\\-11\end{bmatrix}$$

4. (*20 points*): Consider the following set of vectors,

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -8 \\ -7 \\ 5 \\ -3 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 5 \\ 4 \\ -4 \\ 2 \end{bmatrix}, \ \mathbf{v}_4 = \begin{bmatrix} -2 \\ 5 \\ 1 \\ 1 \end{bmatrix}$$

If we form the matrix using these vectors as columns, in the order given, and reduce the matrix to reduced row echelon form we obtain,

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Does this set of vectors form an independent or dependent set?

Solution: Dependent

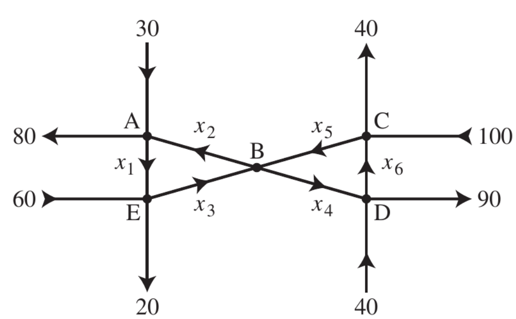
(b) Is any vector in this set a linear combination of the others? If so write the vector as a linear combination of the other vectors. You may use $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ instead of the actual vectors to save time.

Solution: Yes, $\mathbf{v}_4 = 2\mathbf{v}_1 - \mathbf{v}_2 - 2\mathbf{v}_3$

(c) Do these vectors span $\mathbb{R}^4$? Why or why not?

Solution: No, the matrix does not have a pivot in each row.

5. **Extra Credit**: (*10 points*): Given the following network, write the system of linear equations and its corresponding augmented matrix that describes the general flow pattern in the network, **do not solve the system**.



Solution:

$$\begin{aligned} x_2 + 30 &= x_1 + 80 \\ x_3 + x_5 &= x_2 + x_4 \\ x_6 + 100 &= x_5 + 40 \\ x_4 + 40 &= x_6 + 90 \\ x_1 + 60 &= x_3 + 20 \end{aligned} \qquad \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 50 \\ 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -60 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 1 & 0 & -1 & 0 & 0 & 0 & -40 \end{bmatrix}$$