1. (*30 points*): Mark each of the following as either True or False by circling the correct answer.

   (a) **True** **False**: If A is an $m \times n$ matrix, then the range of the transformation $\mathbf{x} \to A\mathbf{x}$ is $\mathbb{R}^m$. **Solution:** False

   (b) **True** **False**: Every linear transformation is a matrix transformation. **Solution:** False

   (c) **True** **False**: When two linear transformations are performed one after another, the combined effect may not always be a linear transformation. **Solution:** False

   (d) **True** **False**: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its effect on the columns of the $n \times n$ identity. matrix. **Solution:** True

   (e) **True** **False**: $AB + AC = A(B + C)$ **Solution:** True

   (f) **True** **False**: Each column of AB is a linear combination of the columns of B using weights from the corresponding column of A. **Solution:** False

   (g) **True** **False**: If A is an invertible $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for *each* $\mathbf{b}$ in $\mathbb{R}^n$ **Solution:** True

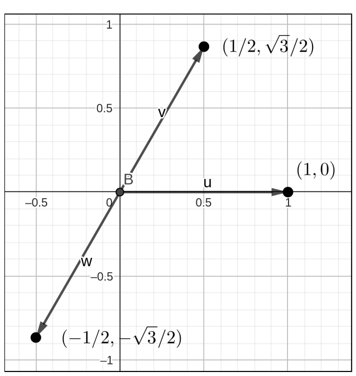
   (h) **True** **False**: Each elementary matrix is invertible. **Solution:** True

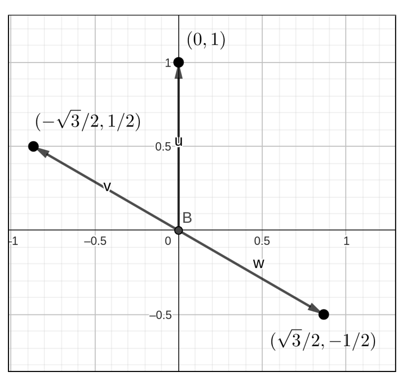
   (i) **True** **False**: If A is an $n \times n$ matrix and the columns of A span $\mathbb{R}^n$, then the columns of A are linearly independent. **Solution:** True

   (j) **True** **False**: If A^T is not invertible, then A is not invertible. **Solution:** True

2. (*15 points*): Find the matrix of the transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that rotates all the vectors counterclockwise by an angle of $\theta = \frac{\pi}{3}$ and then reflects the vector through the origin. Keep your answer in exact form.

   Solution: $A = [T(\mathbf{e_1})\ T(\mathbf{e_2})]$. Start with $\mathbf{e_1} = (1, 0)$ the rotation takes the point to $(1/2, \sqrt{3}/2)$ then the reflection takes that to $(-1/2, -\sqrt{3}/2)$ which is the first column of A. Now $\mathbf{e_2} = (0, 1)$ the rotation takes the point to $(-\sqrt{3}/2, 1/2)$ then the reflection takes that to $(\sqrt{3}/2, -1/2)$ which is the second column of A.





   So the transformation matrix is, $A = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$

Solution: Another way to do this is the rotation counterclockwise by an angle θ is

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

So rotation by $\theta = \frac{\pi}{3}$ is

$$R_{\pi/3} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

Reflection through the origin has matrix,

$$R_{\mathcal{O}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

So the transformation matrix we desire is $R_{\mathcal{O}}R_{\pi/3}$ which is,

$$A = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

3. (*15 points*): Let A and B be as follows, compute AB and BA. If the computation is undefined state why.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 7 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -3 & 2 \\ 1 & -1 & 1 \\ 2 & 5 & -3 \end{bmatrix}$$

Solution:

$$AB = \begin{bmatrix} 0 & -20 & 13 \\ 29 & 9 & -2 \end{bmatrix}$$

BA does not exist since the number of columns of B is not the same as the number of rows of A.

4. (*20 points*): Find the inverse of the following matrix if it exists. If it does not exist, state why.

$$A = \begin{bmatrix} 8 & 3 & 1 \\ -17 & -7 & -2 \\ 10 & 4 & 1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 8 & 3 & 1 & 1 & 0 & 0 \\ -17 & -7 & -2 & 0 & 1 & 0 \\ 10 & 4 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -3 & -2 & -1 \\ 0 & 0 & 1 & 2 & -2 & -5 \end{bmatrix}$$

so the inverse is

$$\begin{bmatrix} 1 & 1 & 1 \\ -3 & -2 & -1 \\ 2 & -2 & -5 \end{bmatrix}$$

5. (*20 points*): Find the LU decomposition of the matrix,

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{bmatrix}$$

Solution:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

6. **Extra Credit**: (*10 points*): Suppose A is an $n \times n$ matrix and the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Let $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}$ show that $A\mathbf{x} = \mathbf{b}$ has a solution.

Solution: If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution then A is an invertible matrix, hence the equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^n$, specifically, $\mathbf{x} = A^{-1}\mathbf{b}$.