1. (*25 points*): Given the following matrix A,

$$A = \begin{bmatrix} -1 & -5 & -1 & -6 & 1 \\ 2 & 10 & 1 & 10 & -3 \\ -1 & -5 & 0 & -4 & 2 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} -1 & -5 & -1 & -6 & 1 \\ 2 & 10 & 1 & 10 & -3 \\ -1 & -5 & 0 & -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 & 4 & -2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis to the column space of A.

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(b) Find a basis to the row space of A.

$$\left\{ \begin{bmatrix} 1 & 5 & 0 & 4 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 2 & 1 \end{bmatrix} \right\}$$

(c) Find a basis to the null space of A.

$$\left\{ \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

2. (*25 points*): Given the following vectors, and $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}\}$ and $\mathcal{C} = \{\mathbf{c_1}, \mathbf{c_2}\}$

$$\mathbf{b_1} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \quad \mathbf{b_2} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \quad \mathbf{c_1} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} \quad \mathbf{c_2} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 7 & -3 & 1 & -2 \\ 5 & -1 & -5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{bmatrix}$$

(a) Find $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$ $\qquad \underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P^{-1}} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$

(b) Find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ $\qquad \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P^{-1}} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$

(c) Given that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{C}}$. $\qquad [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$

3. (*50 points*): Given the following matrix,

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$$

(a) Find the characteristic polynomial of A.
Solution: $x^3 - 7x^2 + 11x - 5$

(b) Find all the eigenvalues of A, and their algebraic multiplicities. Hints: $\lambda = 1$ is an eigenvalue, and all eigenvalues to this matrix are integers.
Solution: $x^3 - 7x^2 + 11x - 5 = (x-5)(x-1)^2$, so $\lambda = 1$ (mult 2), 5 (mult 1).

(c) Find bases to each of the eigenspaces for A.
Solution: For $\lambda = 1$

$$\begin{bmatrix} -1 & -2 & 1 \\ -1 & -2 & 1 \\ 1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \rightarrow \quad \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

For $\lambda = 5$

$$\begin{bmatrix} 3 & -2 & 1 \\ -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \rightarrow \quad \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(d) What is the dimension of each eigenspace and what geometric object is each eigenspace?
Solution: $\dim(E_1) = 2$ (plane through $\mathbf{O}$) and $\dim(E_5) = 1$ (line through $\mathbf{O}$).

(e) Is A diagonalizable? If so, find P and D such that $A = PDP^{-1}$.
Solution: Yes, eigenspace dimensions match algebraic multiplicities for all λ.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad P = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

(f) Is A invertible? Why or why not?
Solution: Yes, 0 is not an eigenvalue of A.

(g) If we define $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ as $T(\mathbf{x}) = A\mathbf{x}$, and we let $\mathcal{B}$ be the set of eigenvectors to A, what is $[T]_{\mathcal{B}}$?
Solution: $[T]_{\mathcal{B}} = D$.

(h) Is T one-to-one? Why or why not?
Solution: Yes, since A is invertible, T is invertible and hence both 1-1 and onto.

(i) Is T onto? Why or why not?
Solution: Yes, since A is invertible, T is invertible and hence both 1-1 and onto.

(j) Is there a plane through the origin in $\mathbb{R}^3$ that T maps to a line through the origin? If so, find a plane that maps to a line, and if not, state why no such plane exists.
Solution: No, since T is 1-1, if such a plane existed an infinite number of points would be mapped to the same point on the image line, which is a contradiction.

4. **Extra Credit**: (*10 points*): The following matrix A was subjected to many iterations of the QR method and the result was the matrix B. Find all the eigenvalues to the matrix A.

$$A = \begin{bmatrix} -\frac{117}{11} & -\frac{54}{11} & -\frac{30}{11} & -\frac{60}{11} & -\frac{21}{11} & -\frac{30}{11} \\ -\frac{90}{11} & \frac{59}{33} & -\frac{10}{33} & \frac{46}{33} & \frac{16}{11} & \frac{56}{33} \\ -\frac{130}{11} & -\frac{455}{33} & -\frac{287}{33} & -\frac{244}{33} & -\frac{60}{11} & \frac{208}{33} \\ \frac{710}{11} & \frac{1000}{33} & \frac{712}{33} & \frac{929}{33} & \frac{150}{11} & \frac{52}{33} \\ \frac{134}{11} & \frac{50}{11} & \frac{18}{11} & \frac{58}{11} & \frac{39}{11} & \frac{40}{11} \\ -\frac{441}{11} & -\frac{878}{33} & -\frac{599}{33} & -\frac{703}{33} & -\frac{124}{11} & \frac{61}{33} \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 7 & 9 & -4 & 3 & 15 \\ 0 & 2 & -1 & -1 & 1 & 6 \\ 0 & 1 & 2 & -5 & 8 & 1 \\ 0 & 0 & 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 & 5 & -4 \\ 0 & 0 & 0 & 0 & 4 & 5 \end{bmatrix}$$

Solution: $\{5,\ -3,\ 5-4i,\ 5+4i,\ 2-i,\ 2+i\}$