Name: ____

Write all of your responses on these pages, use the backs if necessary. Show all your work, answers without supporting justification will not receive credit.

1. Linear Equations: (*45 points*): Consider the following set of vectors in $\mathbb{R}^3$,

$$\mathbf{v}_1 = \begin{bmatrix} 7 \\ -5 \\ -21 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -4 \\ 3 \\ 13 \end{bmatrix}$$

   (a) Write the following vector $\mathbf{w}$ as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, if possible. If it is not possible explain why.

$$\mathbf{w} = \begin{bmatrix} -14 \\ 11 \\ 46 \end{bmatrix}$$

   (b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, linearly independent or dependent? Explain why.

   (c) If we define a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ as $T(\mathbf{x}) = A\mathbf{x}$, where $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$.

      i. Is T one-to-one? Why or why not?

      ii. Is T onto? Why or why not?

   (d) If we define a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ as $T(\mathbf{x}) = B\mathbf{x}$, where $B = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{w}]$.

      i. What are the values of n and m?

      ii. Is T one-to-one? Why or why not?

      iii. Is T onto? Why or why not?

   Note that the next 2 pages are blank for you to continue your solutions to this exercise.

2. Matrix Algebra: (*45 points*): Consider the following set of matrices,

$$A = \begin{bmatrix} -1 & -1 & -3 & 1 & -6 \\ 2 & 1 & 4 & -3 & 5 \\ 5 & 3 & 11 & -7 & 16 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 8 & 7 \\ -9 & -1 & -10 \\ 5 & 7 & -4 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & -4 & -6 \\ 4 & 1 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 5 \\ -8 & 1 \\ 3 & -9 \end{bmatrix}$$

(a) Find the following products if they exist and if not state why.

i. AB

ii. CD

iii. DC

(b) Find a basis to the column space of A.

(c) Find a basis to the row space of A.

(d) Find a basis to the null space of A.

(e) What is the rank of A? What is the nullity of A?

Note that the next 2 pages are blank for you to continue your solutions to this exercise.

3. Vector Spaces: (*40 points*):

   (a) Show that the set $\{c_1 \sin(\omega t) + c_2 \cos(\omega t) \mid c_1, c_2 \in \mathbb{R}, \omega \text{ fixed constant}\}$ is a subspace of $C[0,1]$.

   (b) The set $\mathcal{B} = \{1+t^2, t+t^2, 1+2t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 1 + 4t + 7t^2$ relative to the basis $\mathcal{B}$.

   (c) Given the following vectors, and $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}\}$ and $\mathcal{C} = \{\mathbf{c_1}, \mathbf{c_2}\}$

$$\mathbf{b_1} = \begin{bmatrix} 8 \\ -3 \end{bmatrix} \quad \mathbf{b_2} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \mathbf{c_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \mathbf{c_2} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

      i. Find $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$

      ii. Find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$

Note that the next 2 pages are blank for you to continue your solutions to this exercise.

4. Eigenvalues and Eigenvectors: (*50 points*): Given the following matrix,

$$A = \begin{bmatrix} 0 & -3 & 6 \\ -2 & 1 & 4 \\ -3 & -3 & 9 \end{bmatrix}$$

(a) Find the characteristic polynomial of A.

(b) Find all the eigenvalues of A, and their algebraic multiplicities. Hints: $\lambda = 4$ is an eigenvalue, and all eigenvalues to this matrix are integers.

(c) Find bases to each of the eigenspaces for A.

(d) What is the dimension of each eigenspace and what geometric object is each eigenspace?

(e) Is A diagonalizable? If so, find P and D such that $A = PDP^{-1}$.

(f) Is A invertible? Why or why not?

Note that the next 2 pages are blank for you to continue your solutions to this exercise.

5. Orthogonality: (*20 points*): Given the following set of vectors,

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -8 \\ 6 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 50 \\ 4 \\ -3 \end{bmatrix}$$

(a) Show that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set.

(b) Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$? Why or why not?

(c) Find orthogonal projection of $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ onto $\mathbf{v}_2$ and the component of $\mathbf{w}$ orthogonal to $\mathbf{v}_2$.

(d) Write $\mathbf{w}$ as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if possible, if not state why.

Note that the next page is blank for you to continue your solutions to this exercise.

6. **Extra Credit:** (*10 points*): Given the following matrix A,

$$A = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix}$$

Find matrices P and C such that $A = PCP^{-1}$ and C is the matrix for a rotation and then a scale transformation.
