1. (*25 Points*): Given the two vectors $\mathbf{u} = \langle 2, 3, 4\rangle$ and $\mathbf{v} = \langle 1, -1, 2\rangle$, find

   (a) $\mathbf{u} \cdot \mathbf{v}$
   Solution: 7

   (b) $\mathbf{u} \times \mathbf{v}$
   Solution: $\langle 10, 0, -5\rangle$

   (c) $|\mathbf{u}|$
   Solution: $\sqrt{29}$

   (d) The angle between $\mathbf{u}$ and $\mathbf{v}$.
   Solution: $\cos^{-1}\left(\frac{7}{\sqrt{29}\sqrt{6}}\right) = 1.011407085429384$ rad $= 57.94935736473121°$

   (e) The equation of the plane containing the points $P_1 = (3, 5, 1)$, $P_2 = (5, 8, 5)$, and $P_3 = (4, 4, 3)$. Hint: $\overrightarrow{P_1P_2} = \mathbf{u}$ and $\overrightarrow{P_1P_3} = \mathbf{v}$.
   Solution: $10(x - 3) - 5(z - 1) = 0$

2. (*25 Points*): Given the space curve $\mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

   (a) Set up the integral for finding the length of this curve for $0 \le t \le 1$.
   Solution: $\mathbf{r}'(t) = 2t\mathbf{j} + 3t^2\mathbf{k}$
   $$\int_a^b |\mathbf{r}'(t)|\, dt = \int_0^1 \sqrt{4t^2 + 9t^4}\, dt$$

   (b) Evaluate the integral to find the length of this curve for $0 \le t \le 1$.
   Solution:
   $$\int_0^1 \sqrt{4t^2 + 9t^4}\, dt = \int_0^1 t\sqrt{4 + 9t^2}\, dt = \frac{1}{18}\int_4^{13} \sqrt{u}\, du = \frac{1}{27}u^{3/2}\Big|_4^{13} = \frac{1}{27}(13^{3/2} - 8)$$

3. (*20 Points*): Given the space curve,
$$\mathbf{r}(t) = \left\langle te^{-t}, \frac{t^3 + t}{2t^3 - 1}, t\sin\left(\frac{1}{t}\right)\right\rangle$$

   (a) Find $\lim_{t\to\infty} \mathbf{r}(t)$, if it exists.
   Solution: $\lim_{t\to\infty} \mathbf{r}(t) = \left\langle 0, \frac{1}{2}, 1\right\rangle$

   (b) Find $\mathbf{r}'(t)$.
   Solution:
   $$\begin{aligned}\mathbf{r}'(t) &= \left\langle e^{-t} - te^{-t}, \frac{(2t^3 - 1)(3t^2 + 1) - 6t^2(t^3 + t)}{(2t^3 - 1)^2}, \sin\left(\frac{1}{t}\right) - \frac{1}{t}\cos\left(\frac{1}{t}\right)\right\rangle \\ &= \left\langle e^{-t} - te^{-t}, -\frac{4t^3 + 3t^2 + 1}{(2t^3 - 1)^2}, \sin\left(\frac{1}{t}\right) - \frac{1}{t}\cos\left(\frac{1}{t}\right)\right\rangle\end{aligned}$$

4. (*30 Points*): Given the space curve $\mathbf{r}(t) = \langle 3\sin(t), 2t, 3\cos(t)\rangle$, find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and the curvature.

Solution:

$$\begin{aligned}
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 3\cos(t), 2, -3\sin(t)\rangle}{|\langle 3\cos(t), 2, -3\sin(t)\rangle|} = \frac{1}{\sqrt{13}}\langle 3\cos(t), 2, -3\sin(t)\rangle \\
\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\frac{1}{\sqrt{13}}\langle -3\sin(t), 0, -3\cos(t)\rangle}{|\frac{1}{\sqrt{13}}\langle -3\sin(t), 0, -3\cos(t)\rangle|} = \langle -\sin(t), 0, -\cos(t)\rangle \\
\kappa &= \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\frac{3}{\sqrt{13}}}{\sqrt{13}} = \frac{3}{13}
\end{aligned}$$

5. (*10 Points Extra Credit*): Find the equation for the torsion of $\mathbf{r}(t) = \langle 3\sin(t), 2t, 3\cos(t)\rangle$.

Solution:

$$\begin{aligned}
\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{13}}\langle -2\cos(t), 3, 2\sin(t)\rangle \\
\mathbf{B}'(t) &= \frac{1}{\sqrt{13}}\langle 2\sin(t), 0, 2\cos(t)\rangle \\
\tau &= -\frac{\mathbf{B}'(t) \cdot \mathbf{N}(t)}{|\mathbf{r}'(t)|} = -\frac{-\frac{2}{\sqrt{13}}}{\sqrt{13}} = \frac{2}{13}
\end{aligned}$$