1. (*15 Points*): Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$$

Solution: On the path of $x = 0$

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = 0$$

and on the path of $x = y^3$

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{(x,y)\to(0,0)} \frac{y^3y^3}{y^6+y^6} = \lim_{y\to 0} \frac{y^6}{2y^6} = \frac{1}{2}$$

Hence the limit does not exist.

2. (*20 Points*): Do one and only one of the following:

   (a) Find all the first and second partial derivatives of $f(x, y) = e^{-2x}\cos(y^2)$.
   Solution:

$$\begin{aligned} f_x &= -2e^{-2x}\cos(y^2) \\ f_y &= -2ye^{-2x}\sin(y^2) \\ f_{xx} &= 4e^{-2x}\cos(y^2) \\ f_{yy} &= -2e^{-2x}(\sin(y^2) + 2y^2\cos(y^2)) \\ f_{xy} = f_{yx} &= 4ye^{-2x}\sin(y^2) \end{aligned}$$

   (b) Find an equation of the tangent plane to the surface $z = x\sin(x+y)$ at the point $(-1, 1)$.
   Solution: At $(-1, 1)$, $z = 0$. $f_x = \sin(x+y)+x\cos(x+y)$ and $f_y = x\cos(x+y)$ so at the point $(-1, 1, 0)$ we have $f_x(-1, 1) = \sin(0) - \cos(0) = -1$ and $f_y(-1, 1) = -\cos(0) = -1$. So the equation of the tangent plane is, $z = -(x+1)-(y-1)+0 = -x - y$.

3. (*20 Points*): Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$ when $z = \ln(3x + 2y)$, $x = s\sin(t)$, and $y = t\cos(s)$.

Solution:

$$\begin{aligned} z_x &= \frac{3}{3x+2y} \\ z_y &= \frac{2}{3x+2y} \\ x_s &= \sin(t) \\ x_t &= s\cos(t) \\ y_s &= -t\sin(s) \\ y_t &= \cos(s) \end{aligned}$$

So $\partial z/\partial s = \frac{3}{3x+2y}\sin(t) - \frac{2}{3x+2y}t\sin(s)$ and $\partial z/\partial t = \frac{3}{3x+2y}s\cos(t) + \frac{2}{3x+2y}\cos(s)$.

4. (*20 Points*): Find the directional derivative of $z = x^2e^{-y}$ at $(3, 0)$ in the direction $\mathbf{u} = \langle 3, 4\rangle$. In what direction, on this surface and at that point, would the directional derivative be a maximum?

   Solution: $\nabla f = \langle 2xe^{-y}, -x^2e^{-y}\rangle$, so $\nabla f(3, 0) = \langle 6, -9\rangle$, and $\mathbf{u} = \langle 3, 4\rangle/\sqrt{3^2 + 4^2} = \langle 3/5, 4/5\rangle$. Hence $D_{\mathbf{u}}(f(x, y)) = \nabla f \cdot \mathbf{u} = \langle 6, -9\rangle \cdot \langle 3/5, 4/5\rangle = -18/5$. The direction of the maximum is the direction of $\nabla f(3, 0) = \langle 6, -9\rangle$.

5. (*25 Points*): Do one and only one of the following:

   (a) Find the absolute maximum and minimum values of

   $$f(x, y) = x^2 + xy + y^2 - 6y$$

   on the domain $D = \{(x, y)| - 3 \le x \le 3, 0 \le y \le 5\}$.

   Solution: $f_x = 2x + y$, and $f_y = x + 2y - 6$. So $y = -2x$ and hence $0 = x + 2y - 6 = x - 4x - 6 = -3x - 6$, so $x = -2$ and the corresponding $y = 4$. So our only critical point is $(-2, 4)$ which is in our domain D. Now consider the four curves on the boundary,

   i. When $x = -3$, $f(x, y) = f(-3, y) = 9 - 3y + y^2 - 6y = y^2 - 9y + 9$. This has a critical point at $y = 9/2$, so the point $(-3, 9/2)$ is a point of interest, as are the endpoints $(-3, 0)$ and $(-3, 5)$.

   ii. When $x = 3$, $f(x, y) = f(3, y) = 9 + 3y + y^2 - 6y = y^2 - 3y + 9$. This has a critical point at $y = 3/2$, so the point $(3, 3/2)$ is a point of interest, as are the endpoints $(-3, 0)$ and $(-3, 5)$.

   iii. When $y = 0$, $f(x, y) = f(x, 0) = x^2$. This has a critical value at $x = 0$, so the point $(0, 0)$ is a point of interest, as are the endpoints $(-3, 0)$ and $(3, 0)$.

   iv. When $y = 5$, $f(x, y) = f(x, 5) = x^2 + 5x - 5$. This has a critical value at $x = -5/2$, so the point $(-5/2, 5)$ is a point of interest, as are the endpoints $(-3, 5)$ and $(3, 5)$.

   Evaluating the function at all the points of interest gives the following,

   - $f(-2, 4) = -12$
   - $f(-3, 9/2) = -11.25$
   - $f(3, 3/2) = 6.75$
   - $f(0, 0) = 0$
   - $f(-5/2, 5) = -11.25$
   - $f(-3, 0) = 9$
   - $f(-3, 5) = -11$
   - $f(3, 0) = 9$
   - $f(3, 5) = 19$

   So the absolute maximum 19 at the point $(3, 5)$ and the absolute minimum is -12 at the point $(-2, 4)$.

(b) Find the local maximum and minimum values and saddle point(s) of the function,

$$f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9y$$

Solution:

$$\begin{aligned} f_x &= 3x^2 - 6x \\ f_y &= 3y^2 - 6y - 9 \\ f_{xx} &= 6x - 6 \\ f_{yy} &= 6y - 6 \\ f_{xy} = f_{yx} &= 0 \\ D(x,y) &= (6x-6)(6y-6) = 36(x-1)(y-1) \end{aligned}$$

The solutions to $3x^2 - 6x = 0$ are $x = 0$ and $x = 2$ and the solutions to $3y^2 - 6y - 9 = 0$ are $y = -1$ and $y = 3$. So we have four critical points $(0,-1)$, $(0,3)$, $(2,-1)$, and $(2,3)$. Applying the second derivative test we have,

- At $(0,-1)$, $D(0,-1) > 0$, $f_{xx}(0,-1) < 0$, local maximum.
- At $(0,3)$, $D(0,3) < 0$, saddle point.
- At $(2,-1)$, $D(2,-1) < 0$, saddle point.
- At $(2,3)$, $D(2,3) > 0$, $f_{xx}(2,3) > 0$, local minimum.

(c) Use Lagrange multipliers to find the extreme values of the function $f(x,y) = xy$ subject to the constraint $4x^2 + y^2 = 8$.

Solution: $\nabla f = \lambda \nabla g$, $g(x,y) = 4x^2 + y^2$, and $4x^2 + y^2 = 8$. This gives, $y = 8x\lambda$ and $x = 2y\lambda$. Solving for λ in both equations gives $\lambda = \dfrac{y}{8x}$ and $\lambda = \dfrac{x}{2y}$. So $\dfrac{y}{8x} = \dfrac{x}{2y}$, giving $2y^2 = 8x^2$, or equivalently, $y^2 = 4x^2$. Substituting this into the constraint equation gives $8 = 4x^2 + 4x^2 = 8x^2$, so $x = \pm 1$. Again using the constraint equation, for either value of x we get $y^2 = 4$ and hence $y = \pm 2$. So the points of interest are $(1,2)$, $(1,-2)$, $(-1,2)$, and $(-1,-2)$. Evaluating f at each gives $f(1,2) = 2$, $f(-1,2) = -2$, $f(1,-2) = -2$, and $f(-1,-2) = 2$. So the max is 2 which happens at the two points $(1,2)$ and $(-1,-2)$ and the min is -2 which happens at the two points $(-1,2)$ and $(1,-2)$.

6. (*10 Points*): Do either of the two exercises you did not do in problem #5. Do only one of them.